Masters Exam

Linear Algebra - January, 2017

Do problems 1, 2, and 3.

1. Suppose the set of vectors $\beta \equiv \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for the vector space $\mathcal{V}$.

   (a) Show that the set $\hat{\beta} \equiv \{\vec{v}_1 - \vec{v}_2, \vec{v}_2 - \vec{v}_3, , \ldots, \vec{v}_{n-1} - \vec{v}_n, \vec{v}_n\}$ is a basis for $\mathcal{V}$.

   (b) Find $\hat{\beta}$ from part(a) if $\mathcal{V} = \mathcal{R}^3$ and $\vec{v}_j = \vec{e}_j, j = 1, 2, 3$. Here the vectors $\vec{e}_j, j = 1, 2, 3$ are the standard basis for $\mathcal{R}^3$

$$\beta = \left\{ \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

   (c) The vector $\vec{v} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3$ is called the coordinates of $\vec{v}$ with respect to the base β. Find the coordinates of $\vec{v}$ with respect to $\hat{\beta}$.

2. Let $\mathcal{F}$ denote the vector space of all real valued functions defined on the real line. Further, you **are given** that the solution set $\mathcal{S}$ for the second order differential equation $ay''(t) + by'(t) + cy(t) = 0$ (here, a, b and c are constants) is a two dimensional subspace of $\mathcal{F}$. Assume $b^2 = 4ac$ and $\lambda = -\dfrac{b}{2a}$.

   (a) Show that $y_1(t) = e^{\lambda t}$ is a solution of the differential equation.

   (b) Show that $y_2(t) = te^{\lambda t}$ is a also solution of the differential equation.

   (c) Find a basis for $\mathcal{S}$ and justify your answer.

Theorem: Suppose the set of vectors $\beta \equiv \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for the vector space $\mathcal{V}$. Assume that $L : \mathcal{V} \to \mathcal{W}$ is a linear transformation from $\mathcal{V}$ into the vector space $\mathcal{W}$. If $R(L)$ denotes the range of the transformation L then $R(L) = \text{span}\,\{L\vec{v}_j\}_{j=1}^{n}$.

3. Let $\mathcal{P}_2 = \text{span}\,\{p_j\}$ where $p_j = x^j, j = 0, 1, 2$ and let $\mathcal{M}_{2\times 2} = \text{span}\,\{E_{ij}\}$. Here E_{ij} is the 2×2 matrix with a 1 in the ij position and zeros elsewhere. That is,

$$\begin{aligned} \mathcal{M}_{2\times 2} &= \text{span}\left\{ E_{ij} : \left\{ \begin{array}{c} i = 1,2 \\ j = 1,2 \end{array} \right\}\right\} \\ &= \text{span}\left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \right. \\ &\qquad \left. E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}. \end{aligned}$$

(a) Define $T : \mathcal{P}_2 \to \mathcal{M}_{2\times 2}$ by $T(p) = \begin{bmatrix} p(1) - p(2) & 0 \\ 0 & p(0) \end{bmatrix}$. Find the elements of the set $\{T\vec{p}_j\}_{j=0}^{2}$.

(b) Find a **basis** for the $R(T)$.

(c) Define $S : \mathcal{P}_2 \to \mathcal{M}_{2\times 2}$ by $S(p) = \begin{bmatrix} p(1) - p(2) & p\,'(1) \\ 0 & p(0) \end{bmatrix}$. Find the elements of the set $\{S\vec{p}_j\}_{j=0}^{2}$.

(d) Find a **basis** for $R(S)$.

(e) Are either of the transformations T or S one to one? Justify your answer.

(f) Add a hypothesis to the above **Theorem** so that the conclusion reads

the set $\{L\vec{v}_j\}_{j=1}^{n}$ is a basis for $R(L)$ is valid.