MS Exam: ALGEBRA

Instructions: Work <u>three</u> of the following four problems

1. A group has the property that the square of every element is the identity. Is the group necessarily abelian?

2. Prove that a subgroup of index 2 is normal. Show that a subgroup of index 3 need not be normal.

3. If α is the root of a quadratic equation with integer coefficients, then we can define the "ring extension"

$$Z[\alpha] = \{m + n\alpha : \ m, n \in \mathbf{Z}\}.$$

Define the two rings $R_1 = \mathbf{Z}[\sqrt{5}]$ and $R_2 = \mathbf{Z}\left[\dfrac{1+\sqrt{5}}{2}\right]$.

Show that

- R_1 is a proper subring of R_2.
- There is no ring isomorphism from R_1 onto R_2. Hint: integers cannot map to irrationals, why?)

4. Prove that if M denotes a maximal ideal in the integral domain (commutative with unit) R then the quotient ring R/M is a field.