

Algebra M.S. exam 2004

Notation:

C is the set of complex numbers, Z^+ is the set of positive integers, Q is the set of rational numbers.

1. Let p be a prime and let

$$G := \{z \in C \mid z^{p^n} = 1 \text{ for some } n \in Z^+\}.$$

The group G is the multiplicative group of all p -power roots of unity in C .

- a. Prove that the map $z \rightarrow z^p$ is a surjective homomorphism.
 - b. Show that G is isomorphic to a proper quotient group of itself.
2. Show that $p(x) := x^3 - 2x - 2$ is irreducible in $Q[x]$. Let α be a root of $p(x)$. Find polynomials $r(x)$ and $q(x)$ in $Q[x]$ such that

$$r(\alpha) = (1 + \alpha)(1 + \alpha + \alpha^2) \quad \text{and} \quad q(\alpha) = \frac{1 + \alpha}{1 + \alpha + \alpha^2}.$$

- 3.a. Prove that

$$Q(\sqrt{2} + \sqrt{3}) = Q(\sqrt{2}, \sqrt{3}).$$

- b. Find degree $[Q(\sqrt{2} + \sqrt{3}) : Q]$.
- c. Find an irreducible polynomial satisfied by $\sqrt{2} + \sqrt{3}$.