Algebra M.S. exam 2005

Notation:
$M_2(R)$ is the group of 2×2 matrices over R under multiplication.

1. Consider the matrices

$$R := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, H := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, V := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, D := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, T := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

in $M_2(R)$, and let $G := \{I_2, R, R^2, R^3, H, D, V, T\}$. Here I_2 is the 2×2 identity matrix.

a. Verify that G is a group.

b. Let $G' = \{1, -1\}$ under multiplication be another group. Define $\varphi : G \to G'$ by

$$\varphi(U) = det(U).$$

Is this map a homomorphism? Is it also epimorphism?

c. Describe the factor group G/K where $K = ker\varphi$.

2. Construct fields with a) 8, b) 81 elements. For each field find a generator of the cyclic multiplicative group of nonzero elements.

3.

a. Suppose that degree of the extension $[K : F] = p$ a prime. Show that any subfield E of K containing F is either K or F.

b. Determine the degree of the extension $Q(\sqrt{3 + 2\sqrt{2}})$.

4. Let F be a finite field. Prove that $F[x]$ contains infinitely many primes.