MS Comprehensive Exam: ALGEBRA

Instructions: Please work at most one problem on each side of each sheet of paper, for ease in grading. Thanks. Also, don't quote theorems identical to what you are asked to prove, if there are any such theorems.

(1) Work just <u>one</u> of the following:

   (a) If the order $|G|$ of the finite group G is even, then there exists at least one element $a \in G$ such that $a \neq e$ and $a = a^{-1}$.

   (b) Suppose G is a group of order $|G| = 2p$ with p a prime.. Show that G must have a normal subgroup H of index 2. Recall that the *index* $[G : H]$ is the number of left cosets $\{xH : \ x \in G\}$.

(2) Given the ring $\mathbf{Z}[i] = \{a + bi : \ a, b \in \mathbf{Z}\}$, suppose $I_p := \{a + bi \in \mathbf{Z}[i] : \ p|a \text{ and } p|b\}$, where i denotes the usual imaginary element $i^2 = -1$ in $\mathbf{C}$.

   (a) Show that I_3 is a maximal ideal.

   (b) Show that I_5 is not a maximal ideal. Hint: Define $M = \langle 2 + i \rangle$ to be the ideal generated by $(2 + i)$ in $\mathbf{Z}[i]$.

(3) Show that the following polynomials are irreducible over the given field $\mathbf{F}$.

   (a) $x^2 + 7$ over $\mathbf{R}$.

   (b) $x^3 - 3x + 3$ over $\mathbf{Q}$.

   (c) $x^2 + x + 1$ over $\mathbf{Z}_2$.

   (d) $x^2 + 1$ over $\mathbf{Z}_{19}$.

   (e) $x^3 - 9$ over $\mathbf{Z}_{13}$.

   (f) $x^4 + 2x^2 + 2$ over $\mathbf{Q}$.

(4) Suppose that $\mathbf{F}$ is a subfield of $\mathbf{K}$ with $[\mathbf{K} : \mathbf{F}] = p$, p a prime number.Show that $\mathbf{K} = \mathbf{F}(a)$ for every element $a \in \mathbf{K}$ that is <u>not</u> in $\mathbf{F}$.