Masters Comprehensive Exam
Algebra
January 8, 2007 (1:00-3:00 PM)

$\mathbb{N}$ denotes the natural numbers and $\mathbb{Q}$ denotes the rationals.

1. Suppose that N is a normal subgroup of the finite group G and let $a \in G$. Prove that the order of Na in G/N divides the order of a in G.

2. Let G be a finite group.
   (a) Prove that if there is a homeomorphism of G onto a non-cyclic group of order 4, then there are proper subgroups H, K, L of G with $G = H \cup K \cup L$.
   (b) Prove that if H and K are subgroups of G with $H \cup K = G$ then $H = G$ or $K = G$.

3. Let F be a field, let $p(x) \in F[x]$ be a polynomial with coefficients in F, let $(p(x))$ be the ideal of $F[x]$ generated by $p(x)$, let R be the quotient ring $R = F[x]/(p(x))$, and let $N = \{a \in R : a^n = 0 \text{ for some n} \in \mathbb{N}\}$. Prove that:
   (a) N is an ideal of R.
   (b) $N \neq (0)$ if and only if $p(x)$ is divisible by the square of some positive degree polynomial in $F[x]$.

4. Notation: If K is an extension field of the field F, then $G(K, F) = \{\alpha : \alpha$ is an automorphism of K and $\alpha(a) = a$ for all $a \in F\}$. If H is a subgroup of $G(K, F)$ then $\text{Fix}(H) = \{a \in K : \alpha(a) = a \text{ for all } \alpha \in H\}$.

   Let r be a real root of $x^3 + 2x - 2$, let $F = \mathbb{Q}$, $K = \mathbb{Q}(r)$, and let $H = G(K, F)$. Find $\text{Fix}(H)$.