Masters Comp in Algebra January 2008

$\mathbb{N}$ **= natural numbers** $\mathbb{Z}$ **= integers** $\mathbb{Q}$ **= rationals** $\mathbb{R}$ **= reals**

1. A subgroup H of a group G is said to be characteristic in G if $\varphi(\mathrm{H}) \subseteq H$ for every (surjective) isomorphism $\varphi : G \to G$

   (a) Prove that if H is characteristic in G then H is a normal subgroup of G.

   (b) Prove that the center $Z(G) := \{a \,\varepsilon\, G : ab = ba \text{ for all } b \,\varepsilon\, G\}$ is characteristic in G.

2. Suppose G is a finite abelian group and that $n \,\varepsilon\, \mathbb{N}$ is relatively prime to the order of G. Prove that for each $y \,\varepsilon\, G$ There is an $x \,\varepsilon\, G$ so that $nx = y$.

3. Prove that:

   (a) $\mathbb{Q}$[x] is a principal ideal domain.

   (b) $\mathbb{Z}$[x] is not a principal ideal domain.

   (c) The kernel of the ring homormorphism $\varphi : \mathbb{Z}[\mathrm{x}] \to \mathbb{R}$ that takes x to $1 + \sqrt{2}$ is a principal ideal.

4. Let $f(x) = x^5 - 4x + 2 \,\varepsilon\, \mathbb{Q}[\mathrm{x}]$ and let G be the Galois group of $f(x)$. Prove that:

   (a) $5 \Big| |G|$.

   (b) $2 \Big| |G|$.