Masters Comprehensive Exam - Algebra

January, 2009

CHOOSE THREE

In the following, $\mathbb{Z}$ is the integers and $\mathbb{Q}$ is the rationals.

(1) Let $\mathbf{G}$ be a group and let $\mathbf{D} = \{(g, g) : g \in \mathbf{G}\}$ be the diagonal subgroup of $\mathbf{G} \times \mathbf{G}$. Prove that $\mathbf{D}$ is normal in $\mathbf{G} \times \mathbf{G}$ if and only if $\mathbf{G}$ is abelian.

(2) Suppose that $\mathbf{G}$ is a finite group and $\mathbf{K}$ is a normal subgroup of $\mathbf{G}$. If the index of $\mathbf{K}$ in $\mathbf{G}$ is $\mathbf{m}$ and $\mathbf{m}$ is relatively prime with $|\ \mathbf{K}\ |$, prove that $\mathbf{K} = \{x^{\mathbf{m}} : x \in \mathbf{G}\}$.

(3) Let $\mathbf{R}$ be an integral domain. Call an element $\mathbf{s} \in \mathbf{R}$ **special** if $\mathbf{s} \neq \mathbf{0}$, $\mathbf{s}$ is not a unit, and for each $\mathbf{a} \in \mathbf{R}$ there are $\mathbf{q}, \mathbf{r} \in \mathbf{R}$ with $\mathbf{a} = \mathbf{qs} + \mathbf{r}$ and either $\mathbf{r} = \mathbf{0}$ or $\mathbf{r}$ is a unit. Prove:
- **a:** If $\mathbf{s} \in \mathbf{R}$ is special then the principal ideal $(\mathbf{s})$ is maximal.
- **b:** If $p(x) \in \mathbb{Q}[x]$ and $\deg(p(x)) = 1$, then $p(x)$ is special in $\mathbf{R} = \mathbb{Q}[x]$.
- **c:** There are no special elements in $\mathbf{R} = \mathbb{Z}[x]$.

(4) Let $\mathbf{K}$ be the splitting field of $x^4 - 8x^2 + 15$ over $\mathbb{Q}$.
- **a:** What is $[\mathbf{K} : \mathbb{Q}]$?
- **b:** What is the Galois group of $\mathbf{K}$ over $\mathbb{Q}$?