Masters Comprehensive Exam in Algebra

Week of March 29-April 2 2010

Work every problem and justify your answers.

(1) Recall the definition of the automorphism group $\mathrm{AUT}(G) = \{\varphi : G \to G \mid \varphi \text{ is an isomorphism }\}$. Describe $\mathrm{AUT}(\mathbb{Z}_4)$ and $\mathrm{AUT}(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

(2) Which of the symmetric (permutation) groups S_n, for $n \geq 1$, have a subgroup isomorphic to the dihedral group with 12 elements?

(3) Suppose that $\mathcal{R} = \mathbb{Q}[x]/\langle x^2 - 3x + 2\rangle$. **Prove** any of the following statements that happen to be true, and **disprove** those which are false.
   (a) $\mathcal{R}$ is a field.
   (b) $\mathcal{R}$ is an integral domain.
   (c) $\mathcal{R}$ is a ring isomorphic to a product of two fields.

(4) The roots of an irreducible polynomial $f(x)$ are $\{\alpha, -1/\alpha, \beta, -1/\beta\}$.
   (a) If $\gamma = \alpha - 1/\alpha$, show that $[\mathbb{Q}[\alpha] : \mathbb{Q}[\gamma]] = 2$.
   (b) Show that the Galois group of $f(x)$ has fewer than 24 elements.