

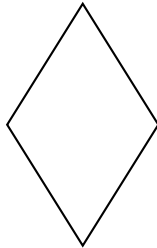
Masters Comprehensive Exam in Algebra
Week of January 3-7, 2011

Work **four** of the following problems, each on a different page, and justify your answers.
It may hurt your score if you attempt more than four problems.

- (1) Let G denote a group containing the distinct elements $\{a, b\}$ and satisfying the following properties:
- (a) Every element in $x \in G$ can be written as a product $x = g_1 g_2 \cdots g_n$, for some n , where each g_k , for $1 \leq k \leq n$, is equal to a or b .
 - (b) $(ab)^2 = ab$.

Is G isomorphic to a familiar group, or is there too little information to decide?

- (2) Let G denote the group of rigid motions (i.e., reflections and rotations) of the diamond shape below.



Answer the following:

- (a) How many elements are in G ?
 - (b) What familiar group is this?
- (3) Find as many (*nonisomorphic*) groups of order 12 as you can.
- (4) Show that $\mathbb{Z} \times \mathbb{Z}$ is isomorphic to a quotient ring of $R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{Z} \right\}$.
- (5) The "Diophantine equation" $x^2 + y^2 = 3z^2$ has the solutions $x = 0$, $y = 0$, and $z = 0$. Show that there are no other *integer solutions* for x, y, z . Hint: First consider solutions in $\mathbb{Z}/4\mathbb{Z}$.
- (6) Find the splitting field and Galois group for $x^4 - 2$ over the rationals.