Sketch of Solutions: Masters Comprehensive Exam in Algebra
Week of January 3-7, 2011

(1) Let G denote a group containing the distinct elements $\{\mathsf{a},\ \mathsf{b}\}$ and satisfying the following properties:
   (a) Every element in $\mathsf{x} \in \mathsf{G}$ can be written as a product $\mathsf{x} = \mathsf{g}_1\mathsf{g}_2 \cdots \mathsf{g}_n$, for some n, where each g_k, for $1 \le k \le n$, is equal to a or b.

   (b) $(\mathsf{ab})^2 = \mathsf{ab}$.

   Is G isomorphic to a familiar group, or is there too little information to decide?

   SOLUTION: $\mathsf{ab} = \mathsf{abab} \implies \mathsf{ab} = 1$ and $\mathsf{b} = \mathsf{a}^{-1}$. We have $1 \ne \mathsf{a} \ne \mathsf{b} \ne 1$. It follows that every element x can be written as a power of a and the group is commutative The map $k \mapsto \mathsf{a}^k$ is a homomorphism of $\mathbb{Z}$ onto G. So G is either infinite cyclic ($\mathbb{Z}$) or finite cyclic ($\mathbb{Z}_n$). NOT ENOUGH INFORMATION.

(2) Let G denote the group of rigid motions (i.e, reflections and rotations) of the diamond shape . Answer the following:
   (a) What is this group? **SOLUTION:** There is a horizontal reflection x and a vertical one y. There is one rotation $z = xy$ by 180 degrees. There is the identity. That's it. The only group of order 4 with no element of order 4 is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(3) Find as many groups of order 12 as you can. **SOLUTION:** Abelian Case: By fundamental theorem of abelian groups $\mathbb{Z}_{12}$ and $\mathbb{Z}_2 \times \mathbb{Z}_6$ are the only possibilities, since there are unique representations $\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2}$ with d_1 dividing d_2. It is not possible to have a third integer d_3 divisible by d_2.

   Nonabelian Case: Certainly the dihedral group $\mathbb{D}_{12}$ and the alternating subgroup $\mathbb{A}_4$ of S_4 are two familiar possibilities. Are there others? Sylow theory will help. $\mathbb{D}_{12} = \mathbb{Z}_2 \times \mathsf{S}_3$ has a unique 3-Sylow subgroup ($n_3 = 1$) and 2-Sylow subgroups of the form $\mathbb{Z}_2^2$ ($n_2 = 3$). The $\mathbb{A}_4$ has $n_2 = 1$ with the 2-Sylow subgroup $\langle (12)(34), (13)(24), (14)(23) \rangle$. Here $n_3 = 4$: $\langle 123 \rangle$, $\langle 124 \rangle$, $\langle 234 \rangle$, and $\langle 134 \rangle$. (Actually if $n_2 = 1$, only the alternating group is possible.) If $n_2 = 3$ and $n_3 = 4$, there is not enough room in the group. If $n_3 = 1$ and $n_2 = 3$, another group is possible besides $\mathbb{D}_{12}$. One proceeds by considering that the 4-groups might be cyclic. One of these 4 groups generates the other two through conjugacy by nontrivial elements of the unique 3-Sylow subgroup. This is enough to find generators and relations for this case. $\mathsf{G} = \langle \mathsf{a}, \mathsf{b} \,|\, \mathsf{a}^3 = \mathsf{b}^2, \mathsf{a}^6 = 1, \mathsf{b}^{-1}\mathsf{ab} = \mathsf{a}^{-1} \rangle$.

(4) Show that $\mathbb{Z} \times \mathbb{Z}$ is isomorphic to a quotient ring of $R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{Z} \right\}$. **SOLUTION:**
   Define the surjective homomorphism $R \to \mathbb{Z} \times \mathbb{Z}$ by $\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\} \mapsto (a, d)$. The kernel is $\left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right\} \cong \mathbb{Z}$.

(5) The "Diophantine equation" $x^2 + y^2 = 3z^2$ has the solutions $x = 0$, $y = 0$, and $z = 0$. Show that there are no other *integer solutions* for $x, y,\ z$. Hint: First consider solutions in $\mathbb{Z}/4\mathbb{Z}$. **SOLUTION:** Squares in $\mathbb{Z}_4$ equal 0 or 1. Assume at least one of x, y, z is nonzero. If each of x, y, and z is even, we can divide 4 out of the Diophantine equation, until one or more of the numbers x, y, z is odd. No such solution is possible mod 4, hence no nontrivial integer solutions exist.

(6) Find the splitting field and Galois group for $x^4 - 2$ over the rationals.

   SOLUTIONS: Factor the polynomial $f(x) = x^4 - 2$ to get $(x^2 - \alpha)(x^2 + \alpha)$ with $\alpha^2 = 2$. A splitting field $\mathbb{K}$ must contain α and a real algebraic number β such that $\beta^2 = \alpha$. Then $(i\beta)^2 = -\alpha$. So K contains β and $i\beta$, and therefore i with $i^2 = -1$. The splitting field is $\mathbb{K} = \mathbb{Q}[\beta, i] = \mathbb{Q}[\beta][i] = \mathbb{Q}[i][\beta]$. Let G denote the galois group of $f(x)$. Since $[\mathbb{K} : \mathbb{Q}] = 8 = |\mathsf{G}|$, G is a subgroup of S_4 with 8 elements. The automorphism σ with $\beta \mapsto i\beta$ and fixing i has order 4. The automorphism τ with $i \mapsto -i$ and fixing β has order 2. The dihedral group $D_8 = \langle \sigma, \tau \rangle$ since $(\sigma\tau)^2 = id$.