ABSTRACT ALGEBRA COMPREHENSIVE EXAM
JANUARY 2015

1. Let G be a group of order $2015 = 5 \cdot 13 \cdot 31$. Show that G has unique normal subgroups of orders 13, 31, and $403 = 13 \cdot 31$, and that all of these are cyclic.

2. Let G be a group with a proper subgroup of finite index. Show that G has a proper normal subgroup of finite index.

3. Let G be a group with presentation $G = \langle a, b \mid a^2 = 1, a^{-1}ba = b^{-1} \rangle$.

(a) Prove that G is infinite. (Hint: One way to do this is to show that every dihedral group D_n is a quotient of G.)

(b) Prove that the commutator subgroup has finite index in G.

4. (a) Find all $a \in \mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ such that the quotient ring $\mathbb{Z}_3[x]/(x^3 + x^2 + ax + 1)$ is a field (with justification.)

(b) In each case in (a), determine how many elements the field has.

5. Let R be a commutative ring with 1 such that every proper ideal is a prime ideal. Show that R is a field. (Hint: In order to show that R is an integral domain, consider the trivial ideal (0). Then, for arbitrary $a \neq 0$ consider the ideal (a^2).)