Real Analysis Master Comprehensive Exam
(August 2015)

Name:

Pick and circle four out of the five problems below, then solve them. If you rely on a theorem please state it carefully! *Good Luck!*

1. Consider the identity map $id : C_{max} \to C_{min}$ where C_{max} is the metric space $C([a,b],\mathbb{R})$ of continuous real valued functions defined on $[a,b]$, equipped with the max metric
$$d_{max}(f,g) = \max_{x\in[a,b]}|f(x) - g(x)|,$$
and C_{int} is $C([a,b],\mathbb{R})$ equipped with the integral metric,
$$d_{int}(f,g) = \int_a^b |f(x) - g(x)|\, dx$$
Prove that id is continuous but its inverse is not continuous.

2. Suppose M and N are metric spaces, that M is covering compact and that $f : M \to N$ is continuous. Use the Lebesgue number lemma to prove that f is uniformly continuous. [**Hint:** Consider the covering of N by $\epsilon/2$-neighborhoods $\{N_{\epsilon/2}(q) : q \in N\}$ and its pre-image in M.]

3. Let (f_n) be a sequence of twice differentiable functions on [0,1] such that for all n, $f_n(0) = 0$ and $f_n'(0) = 0$. Suppose that $|f_n''(x)| \leq 2$ for all n, x. Prove that there is a subsequence of (f_n) which converges uniformly on [0,1].

4. Prove that if the terms of a sequence decrease monotonically, $a_1 \geq a_2 \geq \ldots$, and converge to 0, then the series $\Sigma_{k=1}^{\infty} a_k$ converges if and only if the associated dyadic series
$$a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots = \Sigma_{k=0}^{\infty} 2^k a_{2^k}$$
converges.

5. Let $f : [0,1] \to \mathbb{R}$ be Riemann integrable over $[b,1]$ for every b such that $0 < b \leq 1$. If f is bounded, prove that f is Riemann integrable over $[0,1]$.