MS Real Analysis Examination, August 2017

Instructions: Attempt **five** of the six questions and show all work. Good luck!

1. For part (a) and (b) recall that for a sequence of real numbers (a_n),

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \sup\{a_k : k \geq n\}.$$

(a) Let (x_n) and (y_n) be two bounded sequences of real numbers. Prove that

$$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

(b) True or False:

$$\limsup_{n\to\infty}(x_n y_n) = (\limsup_{n\to\infty} x_n)(\limsup_{n\to\infty} y_n).$$

If true give a proof, if false provide a counter example.

(c) Suppose that $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is a Cauchy sequence.

2. (a) State the Riemann–Lebesgue Theorem on Riemann integration.

(b) Give an example of a function that is not Riemann integrable.

(c) Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/q & \text{if } x \text{ is rational, } x = p/q \text{ in lowest terms and } q > 0 \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Is this function Riemann integrable?

3. Let E be a closed, bounded, and nonempty subset of $\mathbb{R}^m$ and let $f : E \to E$ be a function satisfying $|f(x) - f(y)| < |x - y|$ for all $x, y \in E$, $x \neq y$. Prove that there is one and only one point $x_0 \in E$ such that $f(x_0) = x_0$.

4. Let $f : [0, \infty) \to [0, \infty)$ be a monotonically decreasing function with

$$\int_0^\infty f(x)\mathrm{d}x < \infty.$$

Prove that $\lim_{x\to\infty} x f(x) = 0$.

5. Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous. Calculate the following limits (with proof)

(a)

$$\lim_{n\to\infty} \int_0^1 x^n f(x)\mathrm{d}x.$$

(b)

$$\lim_{n\to\infty} \int_0^{\frac{1}{2}} f(x^n)\mathrm{d}x.$$

6. Let (f_n) be a sequence of uniformly bounded real valued continuous functions on $[0,1]$ such that for every natural $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \int_0^1 f_n(x)x^k \mathrm{d}x = 0.$$

Show that for every continuous $\phi : [0,1] \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)\phi(x) \mathrm{d}x = 0.$$