Problem 1. Suppose that E is an uncountable subset of $\mathbb{R}$. Let the set M be defined by

$$M = \{x \in \mathbb{R} \mid (-\infty, x] \cap E \text{ is finite or countable}\}.$$

Prove that if $M \neq \emptyset$, then M is bounded above Also show that if $b = \sup M$, then $b \in M$, and for some x, $x > b$, both sets $(-\infty, x] \cap E$ and $[x, +\infty) \cap E$ are uncountable, still assuming that $M \neq \emptyset$.

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Problem 2. Suppose that $\{a_n\}$ is a decreasing sequence of non-negative real numbers: $a_n \geq a_{n+1} \geq 0$ for $n = 1, 2, \ldots$. Prove that if $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} na_n = 0$.

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Problem 3. Suppose that $f : [0, +\infty) \to \mathbb{R}$ is a continuous function such that $\lim_{x \to +\infty} f(x) = 0$. Prove that $\lim_{x \to +\infty} \frac{1}{x} \int_0^x f(t)\, dt = 0$.

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Problem 4. Suppose that $\{f_n\}$ is a sequence of continuous functions on the interval $[0,1]$ which converges pointwise to 0 on $[0,1]$; that is for each $x \in [0,1]$

$$\lim_{n\to\infty} f_n(x) = 0.$$

Show that if, for each $x \in [0,1]$, the sequence $\{f_n(x)\}$ is decreasing, then $\{f_n\}$ converges uniformly to 0 on the interval $[0,1]$.

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