ANALYSIS: Master's Comprehensive Exam January '06

Instructions: Attempt all of the problems, showing work. Place at most one problem solution on a side for each sheet of paper turned in. Do not submit your scratch work.

(1) If $0 < x < \frac{\pi}{2}$, prove that $\frac{2}{\pi} < \frac{\sin x}{x} < 1$.

(2) Suppose that $f : [0, \infty) \to \mathbf{R}$ is continuous and $\lim_{x\to\infty} f(x) = 1$.

(a) For $0 < R < \infty$, evaluate $\lim\limits_{n\to\infty} \frac{1}{n} \int_0^R e^{-\frac{x}{n}} f(x)\, dx$

(b) Show $\lim\limits_{n\to\infty} \frac{1}{n} \int_0^\infty e^{-\frac{x}{n}} f(x)\, dx = 1$

(3) Let (f_n) denote a sequence of continuous functions on a domain $D \subset \mathbf{R}$ such that f_n converges uniformly to the function f on D.

Show that $\lim\limits_{n\to\infty} f_n(x_n) = f(x)$, if $\lim\limits_{n\to\infty} x_n = x$ and $x \in D$.