ANALYSIS: Master's Comprehensive Exam

April 5, 2006

Instructions: Attempt all of the problems, showing work. Place at most one problem solution on a side for each sheet of paper turned in. Do not submit your scratch work.

(1) In each case below, give examples of pairs of sequences $\{x_n, y_n\}$ such that $x_n \to 0$ and $y_n \to \infty$ and which satisfy the specified limits as $n \to \infty$. Prove that your examples are really correct.
   (a) $x_n y_n \to 0$
   (b) $x_n y_n \to 6$
   (c) $x_n y_n$ is bounded but has no limit.

(2) Suppose that second derivative $f''(x) < 0$ on (a, b), $a < b$, and the derivative f' is continuous on $[a, b]$. Prove that $(b, f(b))$ must lie under the tangent line to the graph at the point $(a, f(a))$.

(3) Given the sequence of functions

$$f_{nk}(x) = \frac{2n^k x}{(1 + n^2x^2)^2} \qquad n = 1, 2, \ldots,$$

defined for $0 \le x \le 1$ and $k \in \mathbf{R}$,
   (a) for what k does f_{nk} converge uniformly on $[0, 1]$ and what is the uniform limit $f = \lim_{n\to\infty} f_{nk}$?
   (b) Find an antiderivative $I_{nk}(x) = \int f_{nk}(x)\, dx$, for all k, n,
   (c) For what k values does $\lim_{n\to\infty} \int_0^1 f_{nk}(x)\, dx$ exist? Compare with the answer to Problem(3a).