Masters Comprehensive Exam
Analysis
January 8, 2007 (9:00-11:00 AM)

CHOOSE 4 Clearly indicate your choices (we will only grade 4 problems).

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable everywhere and that $f''(x) \neq 0$ for all x. Prove that f can have at most one local extremum.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Prove that
$$\lim_{n\to\infty} n \int_0^1 f(x + \frac{1}{n}) - f(x)dx = f(1) - f(0).$$

3. Prove that $f_k(x) = \sum_{n=0}^{\infty} x^n \cos(\frac{n\pi x}{k})$ defines a continuous function on the interval $-1 < x < 1$, for each $k = 1, 2, \cdots$, and compute (with justification) $\lim_{k\to\infty} \int_0^{1/2} f_k(x)dx$.

4. Suppose that, for $n = 1, 2, \cdots$, $f_n : \mathbb{R} \to \mathbb{R}$ satisfies:

   (a) $|f_n(x) - f_n(y)| \leq 2007\,|x - y|$ for all x, y; and

   (b) $f_n(0) = 0$.

   Prove that some subsequence of $\{f_n\}$ converges uniformly to a Lipschitz function on $[0, 100]$.

5. Prove that there is a C^1 function $g : U \to \mathbb{R}$, U a neighborgood of $(0, 0)$ in $\mathbb{R}^2$, that satisfies:

   (a) $g(x_1, x_2) = 3x_1 + 2x_2 + \int_0^{g(x_1,x_2)} \frac{3}{(1 + t^2)^{1/3}}\, dt$ for all $(x_1, x_2) \in U$; and

   (b) $g(0, 0) = 0$.

   Find the derivative (differential) $dg(0, 0)$.