MASTER'S EXAM IN ANALYSIS

January 2008

1. Let X be a metric space and let $f : X \to \mathbb{R}$.

   (a) Prove that there exists a sequence $\{x_n\}$ with $x_n \in X$ for which
   $$\lim_{n\to\infty} f(x_n) = \inf_{x\in X} f(x).$$

   (b) If f is continuous and X is compact, prove that $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges to some $x_* \in X$ for which $f(x_*) = \inf_{x\in X} f(x)$.

   (c) Provide a counterexample to show that the conclusion in part (b) need not hold if f is not continuous but X is compact.

   (d) Provide a counterexample to show that the conclusion in part (b) need not hold if X is not compact but f is continuous.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x_0 \in (a, b)$. Prove that if $f(x_0) \le f(x)$ whenever $a < x < b$, then $f'(x_0) = 0$. Then state and prove an analogous result for $f : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^{\mathrm{n}}$ is open.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on the interval $[a, b]$, $-\infty < a < b < \infty$.

   a. Prove that f is Riemann integrable on $[a, b]$.

   b. Provide an example of a function f which is Riemann integrable but not continuous on $[a, b]$.

   c. Provide an example of a function which is not Riemann integrable on the interval $[0, 1]$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ have the series representation
   $$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^3} \sin(nx)$$

   a. Prove that this series converges uniformly on $\mathbb{R}$.

   b. Is f continuous? Is so, prove it; otherwise explain why not.

   c. Is f uniformly continuous on $\mathbb{R}$? If so, prove it; otherwise explain why not.

   d. Provide a series representation for the derivative $f'(x)$, and prove that this series does indeed converge to $f'(x)$ and that the convergence is uniform.