Masters Comprehensive Exam - Analysis

January 7, 2009

CHOOSE THREE

(1) Suppose that $f : [1,2] \to \mathbb{R}$ is continuous. Find (with justification) the limits

$$(a) \qquad \lim_{n\to\infty} \int_1^2 x^{-n} f(x)dx,$$

$$(b) \qquad \lim_{n\to\infty} \int_1^2 nx^{-n} f(x)dx.$$

(2) Prove that $\lim_{x\to 1/2} \sum_{k=0}^{\infty} x^k \cos(k\pi x) = 4/5$

(3) Given $f \in \mathbf{C}^0[-1/2, 1/2]$, let $\mathbf{K}f \in \mathbf{C}^0[-1/2, 1/2]$ be defined by

$$\mathbf{K}f(x) = \int_0^x (x+t)\sin(f(t))dt.$$

Prove that there is a unique $f \in \mathbf{C}^0[-1/2, 1/2]$ such that $\mathbf{K}f = f$.

(4) Suppose that (f_n) is a sequence in $\mathbf{C}^0[a,b]$ with the properties:
(i) f_n is differentiable on (a,b) for all n,
(ii) there is a $\mathbf{B} < \infty$ such that $| f_n'(x) | \leq \mathbf{B}$ for all n and all $x \in (a,b)$,
(iii) $(f_n(a))$ is convergent.
Prove that some subsequence of (f_n) converges uniformly on $[a,b]$.