Problem 1. Suppose that E is a bounded subset of $\mathbb{R}$ and that $f : E \to \mathbb{R}$ is uniformly continuous. Prove that f is bounded.

Problem 2. Suppose that $f : [0, \frac{1}{2}] \to \mathbb{R}$ is continuous. Calculate (with justification) $\lim_{n\to\infty} \int_0^{\frac{1}{2}} f(x^n)\, dx$.

Problem 3. Prove that

$$\lim_{x\to 0} \frac{\int_x^{2x} \sum_{n=1}^{\infty} \frac{\sin(nt)}{n^3 t}\, dt}{x} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Problem 4. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, that $\frac{\partial f}{\partial y}(x, y)$ exists for all $(x, y) \in \mathbb{R}^2$, and that $\left|\frac{\partial f}{\partial y}(x, y)\right| \leq \frac{1}{2}$ for all $(x, y) \in \mathbb{R}^2$. Prove that there is a unique continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying $g(x) = f(x, g(x))$ for all $x \in \mathbb{R}$.