Real Analysis Master Comprehensive Exam
(Jan 2015)

Name:

Pick and <u>circle</u> four out of the five problems below, then solve them. If you rely on a theorem please state it carefully! *Good Luck!*

1. Construct a series $\sum_{k=0}^{\infty} a_k$ such that $(-1)^k a_k > 0$ for all k, $a_k \to 0$ as $k \to \infty$, but the series diverges.

2. Let (x_n) be a sequence in $\mathbb{R}$. Prove that (x_n) has a monotone subsequence.

3. Let M be a compact metric space and A be a dense subset of M. Prove that for any $\delta > 0$, there exists a finite subset $\{a_1, \cdots, a_k\} \subset A$ which is **δ-dense** in M in the sense that each $x \in M$ lies within distance δ of at least one of the points $a_j, j = 1, \cdots, k$.

4. Let (f_n) be a sequence of differentiable functions defined on $[0,1]$, and assume that for all n, $f_n(0) = f_n'(0)$. Suppose also that for all n and $x \in [0,1]$, $|f_n'(x)| \leq 1$. Prove that there is a subsequence of (f_n) converges uniformly on [0,1].

5. Let $f(x) : [0,1] \to \mathbb{R}$, be a continuous function with continuous derivative $f'(x)$, and there exists a constant $M > 0$ such that $|f'(x)| \leq M$ for all $x \in [0,1]$. Prove that for any $n \in \mathbb{N}$,

$$\left| \frac{1}{n} \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) - \int_0^1 f(x)\, dx \right| \leq \frac{M}{2n}$$