M.S. COMPREHENSIVE EXAMINATION IN APPLIED MATHEMATICS
April 2, 2001

1. A small pond is stocked with game fish, both trout and bass. Let $x(t)$ denote the population of the trout in the pond at any time t and let $y(t)$ denote the population of the bass in the pond at any time t. Denote the fundamental dimensions as $[x] = [y] = P$, and $[t] = T$. Assume that without the bass, the trout population could grow indefinitely (growth rate proportional to the population x with proportionality constant k_1). However the bass population decreases the growth rate of the trout population in a manner proportional to the number of possible interactions between the two species, say proportional to the product of x and y with proportionality constant k_2. A similar situation is assumed for the growth of the bass population, where the proportionality constant for the interaction term is k_2 again. If the trout population is initially x_0 and the bass population is initially y_0, a mathematical model that describes $x(t)$, the trout population, and $y(t)$, the bass population, at time $t \geq 0$ is given by

$$\begin{aligned} \frac{dx}{dt} &= k_1 x - k_2 xy, \quad x(0) = x_0 \\ \frac{dy}{dt} &= k_3 y - k_2 xy, \quad y(0) = y_0 \end{aligned}$$

(a) Use dimensional analysis to determine the fundamental dimensions of all the variables and parameters. Identify the fundamental dimensions of x, y, t, k_1, k_2, k_3, x_0, and y_0.

(b) Determine how many independent dimensionless variables and parameters are necessary to describe this physical process.

(c) Scale the problem to arrive at a dimensionless model involving dimensionless populations X and Y and dimensionless time s. Carefully identify the dimensionless parameters.

2. Consider the nonlinear initial-value problem

$$\begin{aligned} y''(x) + y'(x) + \varepsilon y^2 &= 0 \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

where ε is a small positive parameter.

(a) Carefully identify the leading order and first-order initial-value problems to be solved for a perturbation solution.

(b) Solve the two initial-value problems above.

(c) Give the two-term perturbation solution which involves the leading order approximation and the first-order correction.

3. Consider the isoperimetric problem

$$J(y) = \int_0^1 xy(x)dx$$

subject to the constraint

$$W(y) = \int_0^1 (y(x))^2 dx = \frac{1}{12} \ .$$

Define the functional $J^*(y) = J(y) + \lambda W(y)$ and the class of admissable functions $A = \left\{ y \in C[1,2] : W(y) = \frac{1}{12} \right\}$.

(a) Write down the Euler equation for this problem and show that the extremals in A are $y_0(x) = \pm x/2$.

(b) Show that $J(y_0 + y) = J(y_0) + J(y)$.

(c) Show that if $y_0 + y$ is in the admissable class A then $W(y_0 + y) = W(y_0) + W(y) \pm J(y)$ for either extremal y_0.

(d) Beginning with $y_0 = \pm x/2 \in A$, assume that y is such that $y_0 + y$ is in A. Then $W(y_0) = W(y_0 + y) = \frac{1}{12}$. Use this and part (c) to show that $W(y) = \mp J(y)$.