M.S. COMPREHENSIVE EXAMINATION IN APPLIED MATHEMATICS
January 14, 2003

1. Use singular perturbation techniques to find the leading order uniform approximation to the solution to the ODE boundary value problem

$$\begin{aligned} \epsilon\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y^3 &= 0, \quad 0<t<1, \\ y(0) &= 0, \\ y(1) &= \frac{1}{2}. \end{aligned}$$

2. Find the exremal for the functional

$$J(y)=\frac{1}{2}\int_0^1 y'(x)^2\,dx+\frac{1}{2}y(0)^2+y(0)$$

subject to the constraints $y\in C^2[0,1]$ and $y(1)=2$.

3. Consider the wave equation on an infinite interval given by

$$\begin{aligned} u_{tt} &= c^2u_{xx}\ , \quad -\infty<x<\infty\ , \quad t>0 \\ u(x,t) &\to 0\ , \quad |x|\to 0 \\ u_x(x,t) &\to 0\ , \quad |x|\to 0 \\ u(x,0) &= f(x)\ , \quad -\infty<x<\infty \\ u_t(x,0) &= 0\ , \quad -\infty<x<\infty\ , \end{aligned}$$

where f is assumed to have a Fourier transform. Solve this using Fourier transforms. Hint: The following shift formula for the function g (assumed to have a Fourier transform) may be helpful.

$$\mathcal{F}\{g(x-a)\}=e^{-i\xi a}\mathcal{F}\{g(x)\}\ .$$

Hence

$$(\mathcal{F}g)(\xi)\cos(c\xi t)=(1/2)\mathcal{F}(g(x))[e^{-ic\xi t}+e^{ic\xi t}]$$

leads to the fact that the inverse Fourier transform for $(\mathcal{F}g)(\xi)\cos(c\xi t)$ is given by $(1/2)[g(x-ct)+g(x+ct)]$.