

M.S. COMPREHENSIVE EXAMINATION IN APPLIED MATHEMATICS
January 14, 2003

1. Use singular perturbation techniques to find the leading order uniform approximation to the solution to the ODE boundary value problem

$$\begin{aligned}\epsilon \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + y^3 &= 0, \quad 0 < t < 1, \\ y(0) &= 0, \\ y(1) &= \frac{1}{2}.\end{aligned}$$

2. Find the extremal for the functional

$$J(y) = \frac{1}{2} \int_0^1 y'(x)^2 dx + \frac{1}{2} y(0)^2 + y(0)$$

subject to the constraints $y \in C^2[0, 1]$ and $y(1) = 2$.

3. Consider the wave equation on an infinite interval given by

$$\begin{aligned}u_{tt} &= c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0 \\ u(x, t) &\rightarrow 0, \quad |x| \rightarrow 0 \\ u_x(x, t) &\rightarrow 0, \quad |x| \rightarrow 0 \\ u(x, 0) &= f(x), \quad -\infty < x < \infty \\ u_t(x, 0) &= 0, \quad -\infty < x < \infty,\end{aligned}$$

where f is assumed to have a Fourier transform. Solve this using Fourier transforms. Hint: The following shift formula for the function g (assumed to have a Fourier transform) may be helpful.

$$\mathcal{F}\{g(x - a)\} = e^{-i\xi a} \mathcal{F}\{g(x)\}.$$

Hence

$$(\mathcal{F}g)(\xi) \cos(c\xi t) = (1/2)\mathcal{F}(g(x))[e^{-ic\xi t} + e^{ic\xi t}]$$

leads to the fact that the inverse Fourier transform for $(\mathcal{F}g)(\xi) \cos(c\xi t)$ is given by $(1/2)[g(x - ct) + g(x + ct)]$.