APPLIED MATH MASTER'S EXAM

January 2007

1. Use standard techniques (i.e., not the Poincare-Lindstedt method) to find a two-term regular perturbation approximation to the solution of the ODE IVP.

$$\frac{d^2y}{dt^2} + \epsilon\frac{dy}{dt} + y = 0, \qquad t > 0$$
$$y(0) = 1, \ \frac{dy}{dt}(0) = 0, \quad 0 < \epsilon \ll 1.$$

2. Give the boundary value problem which characterizes the extremal for the functional

$$J(y) = \frac{1}{2}\int_0^1 \left[\left(\frac{dy}{dx}\right)^2 + y(x)^2\right] dx - 2y(1)$$

subject to the constraints $y \in C^2[0,1]$ and $y(0) = 0$.

3. Show that $f(\epsilon) = O(g(\epsilon))$, where $\epsilon \to 0^+$,

$$f(\epsilon) = \int_0^\epsilon e^{-x^2}\,dx \qquad \text{and} \qquad g(\epsilon) = \epsilon.$$

4. (a) Compute the eigenvalues λ_n and corresponding eigenfunctions $\phi_n(x)$ for the SLP

$$-\phi'' = \lambda\phi, \qquad \phi(0) = 0, \ \phi(\pi) = 0.$$

   (b) Then obtain a series representation for the solution to the boundary value problem

$$-u'' + \alpha^2 u = f(x), \quad 0 < x < \pi,$$
$$u(0) = 0, \ \ u(\pi) = 0$$

   Note that $\int_0^1 \sin^2(nx)\,dx = \pi/2$.

5. Let $D(a,b)$ denote the set of test functions on the interval (a,b), and let $D'(a,b)$ denote the set of distributions (continuous linear functionals) on the set $D(a,b)$.

   (a) Does $(u,\phi) = [\phi(0)]^2 \ \forall\ \phi \ \in D(-1,1)$ define a distribution in $D'(-1,1)$?

   (b) Let $H(x)$ denote the Heaviside function, and let $\delta(x)$ denote the Dirac Delta function with pole at 0. Show that the $\delta(x)$ is the distributional derivative of $H(x)$. That is, show that $H'(x) = \delta(x)$ in the sense of distributions.