APPLIED MATH MASTER'S EXAM

January 2010

Instructions: Attempt 4 of the following 5 questions. Show all work. Carefully Read and follow the directions. Clearly label your work and attach it to this sheet.

1. A collection of dominos have height h and thickness d. The dominos are set upright and are equally spaced by a distance $t = d$ (the same as the domino thickness). After the dominos are set in motion they eventually achieve a terminal velocity v. Letting g be the gravitational constant we assume that there is a physical law relating these dimensional quantities. Specifically, we assume there is a function f such that

$$f(d, h, v, g) = 0$$

   a) Find the two dimensionless quantities π_1 and π_2 for this law having the form.

$$\pi = d^{\alpha_1} h^{\alpha_2} v^{\alpha_3} g^{\alpha_4}$$

   b) Use your results from a) and the Buckingham-Pi Theorem to then determine the terminal velocity v in terms of the other dimensional parameters.

2. Let $y(x, \epsilon)$ be the solution of the boundary value problem:

$$\begin{aligned} \epsilon y'' + (x+1)\ y' + y &= 0 \quad , \quad x \in (0,1) \\ y(0) &= 0 \quad , \quad y(1) = 1 \end{aligned}$$

   Find a uniformly valid approximation $y_u(t, \epsilon)$ in the limit $\epsilon \to 0$ assuming a layer exists at $x = 0$.

3. A functional J is defined on $\mathcal{A}$ where

$$\begin{aligned} J(y) &= -\frac{1}{y(1)} + \int_1^e xy'(x)^2\ dx \\ \mathcal{A} &= \left\{ y \in C^2[1, e]\ : y(e) = 0 \right\} \end{aligned}$$

   a) Derive the natural boundary condition for $y(x)$.

   b) Find the extrema $\bar{y}(x)$ of J over $\mathcal{A}$.

4. Let $c > 0$ and $g > 0$ both be positive constants. Solve the following initial value problem **using Laplace Transforms** assuming the solution $u(x,t)$ is bounded in x.

$$u_{tt} - c^2 u_{xx} = -g, \quad x > 0,\ t > 0$$

$$u(0,t) = 0, \quad t > 0$$

$$u(x,0) = 0,\ \ u_t(x,0) = 0, \quad x > 0.$$

HINT: You may use the following relation.

$$\mathcal{L}\{H(t-a)f(t-a)\} = F(s)e^{-as}$$

where $\mathcal{L}\{f(t)\} = F(s)$, and $H(t)$ denotes the Heaviside function.

5. Find the eigenvalues and eigenfunctions for the following Sturm-Liouville problem

$$-y''(x) = \lambda y(x), \quad 0 < x < \ell$$

$$y'(0) = 0,\ \ y(\ell) = 0$$

with mixed Dirichlet and Neumann boundary conditions.