Applied Mathematics Comprehensive Exam

August 2012

Instructions: Answer 3 of the problems from **Part A**, and answer 3 of the problems from **Part B**. Indicate clearly which questions you wish to be graded.

Part A

A.1 (a) Find the Singular Value Decomposition, $A = U\Sigma V^T$, where

$$A = \begin{bmatrix} 2\sqrt{5} & -2\sqrt{5} \\ 3 & 3 \\ 6 & 6 \end{bmatrix}$$

(b) Use your answer in part (a) to compute the pseudoinverse of A.

(c) Find conditions on $\vec{y}$ for which the system

$$A\vec{x} = \vec{y}$$

has a solution (where A is as given above).

(d) Find the least squares solution of $A\vec{x} = \vec{b}$ with A given above and

$$\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

A.2 (a) State in detail the Fredholm Alternative Theorem for

$$u(x) = f(x) + \lambda \int_a^b k(x,y)u(y)dy$$

(b) For what values of λ does the following integral equation have solutions? Identify solvability conditions when they exist.

$$u(x) = f(x) + \lambda \int_0^{2\pi} \sum_{j=1}^{n} \frac{1}{j} \cos(jx)\cos(jy)u(y)dy$$

A.3 For the boundary value problem given below,

$$u'' + \lambda u = f(x), \quad 0 < x < 1$$
$$u(0) = 1, \quad u'(1) = 0$$

(a) Identify all values of λ where a Green's function exists.

(b) For the specific case of $\lambda = 0$, compute the Green's function, and use it to construct the solution of the given boundary value problem.

A.4 (a) Find solvability conditions for

$$u'' = \sin^2 x, \quad 0 < x < \pi$$
$$u'(0) = \alpha, \quad u'(\pi) = \beta$$

(b) Solve the boundary value problem in the least squares sense.

Part B

B.1 (a) Define what is meant by a test function and a distribution.

(b) Let $f(x) = |x|$ be defined on $\mathbb{R}^1$. Find the distributional derivatives f' and f'' on $\mathbb{R}^1$. Show your work.

B.2 Find a modified Green's function for

$$u'' = f(x)$$

$$u(0) = u(1), \quad u'(0) = u'(1)$$

B.3 Use an appropriate eigenfunction expansion to represent the solution (when it exists) of the given problem. Explicitly state the solvability condition on $f(x)$.

$$u'' = f(x), \; 0 < x < 1$$

$$u'(0) = 0, \quad u'(1) = 0$$

B.4 Consider the variational problem

$$\min \int_a^b F(x, y(x), y'(x))\, dx$$

over the space of functions y that are piecewise continuously differentiable on $[a, b]$ with $y(a) = 0$ and $y(b)$ unspecified. Assume that $F(x, y, y')$ is a smooth function. Derive the Euler-Lagrange equation for this problem, and clearly identify the appropriate boundary conditions. Be sure to describe the appropriate space of admissible variations.

B.5 Use Neumann iterates to solve the following integral equation using $u_0(x) = 1$.

$$u(x) = 1 + \int_0^x (y - x)u(y)dy$$