

APPLIED MATH MASTER'S EXAM

January 2013

Instructions: Attempt 4 of the following 6 questions. Show all work. Carefully read and follow the directions. Clearly label your work and attach it to this sheet.

1. Two planets of mass m_1 and m_2 orbit each other under their mutual gravitational attraction. The motion is periodic with period T and their mean separation (distance) is r . The universal gravitational constant G for such motion is $G = 6.67 \times 10^{-11} m^3 kg^{-1} sec^{-2}$. We assume there is a physical law relating the aforementioned quantities. Specifically, we assume there is a function f such that:

$$f(r, m_1, m_2, T, G) = 0$$

- a) Find the two dimensionless quantities π_1 and π_2 for this law having the form:

$$\pi = T^{\alpha_1} r^{\alpha_2} m_1^{\alpha_3} m_2^{\alpha_4} G^{\alpha_5}$$

- b) Use your results from a) and the Buckingham-Pi Theorem to then determine the period T in terms of the other dimensional parameters.

2. Let $y(x, \epsilon)$ be the solution of the boundary value problem:

$$\begin{aligned} \epsilon y'' + y' + xy^3 &= 0 \quad , \quad x \in (0, 1) \\ y(0) &= 0 \quad , \quad y(1) = \frac{1}{\sqrt{5}} \end{aligned}$$

Find a uniformly valid approximation $y_u(x, \epsilon)$ in the limit $\epsilon \rightarrow 0$ assuming a layer exists at $x = 0$.

3. A functional J is defined on \mathcal{A} where

$$J(y) = \int_0^1 \frac{1}{(y')^2 + 1} dx$$

$$\mathcal{A} = \{y \in C^2[0, 1] : y(1) = 7\}$$

- a) Derive the natural boundary condition for $y(x)$.
- b) Find the extrema $\bar{y}(x)$ of J over \mathcal{A} . Is $\bar{y}(x)$ a min, max or neither?

4. Find the eigenvalues and eigenfunctions for the following Sturm-Liouville problem:

$$-y''(x) = \lambda y(x) \quad , \quad 0 < x < \pi$$

$$y(0) - y'(0) = 0, \quad , \quad y(\pi) - y'(\pi) = 0$$

5. Find the Green's function for the following problem

$$-\frac{d}{dx} \left(x \frac{du}{dx} \right) = f(x) \quad , \quad x \in (1, e)$$

$$u(1) = 0$$

$$u(e) = 0$$

6. Find the general solution of

$$u_{xt} + u_t = t^2 \quad , \quad x \in \mathbb{R} \quad , \quad t > 0$$

and then use it to find the solution of the boundary value problem:

$$u_{xt} + u_t = t^2$$

$$u(x, 0) = x \quad , \quad x > 0$$

$$u(0, t) = t \quad , \quad t > 0$$