APPLIED MATH MASTER'S EXAM

January 2014

Instructions: Attempt 4 of the following 6 questions. Show all work. Carefully read and follow the directions. Clearly label your work and attach it to this sheet.

1. The following problem involves the impact deformation of an elastic ball as it hits a wall. Suppose a ball of diameter D, and density ρ is travelling velocity v just before it strikes a wall. The elastic properties of the ball is determined by the modulus of elasticity E where $[E] = [\rho v^2]$. After the ball hits the wall it has a deformed diameter d.

   a) Use the Buckingham Π-Theorem to determine the <u>deformed</u> diameter d as a function of all other quantities assuming the law

   $$f(D, d, \rho, V, E) = 0$$

   for some function f.

   b) Two balls made of different materials have the same diameter D and density ρ. The first ball has a deformed diameter d at speed v whereas the second has the same deformed diameter d at speed $2v$. What is the ratio of the balls elastic moduli?

2. Let $y(x, \epsilon)$ be the solution of the singular boundary value problem:

   $$\begin{aligned} \epsilon y'' + \ y' + xy^2 &= 0 \quad , \quad x \in (0,1) \quad , \quad 0 < \epsilon \ll 1 \\ y(0) &= A \quad , \quad y(1) = 1 \end{aligned}$$

   Find a uniformly valid approximation $y_u(x, \epsilon)$ when $A = 1$ assuming a layer exists at $x = 0$. For what value of A is there no layer?

3. A functional J is defined on $\mathcal{A}$ where

   $$J(y) \ = \ y\left(\frac{\pi}{4}\right)^2 + \int_0^{\pi/4} \cos^2 x \ y'^2 \ dx$$

   $$\mathcal{A} \ = \ \left\{ y \in C^2[0, \pi/4] \ : y(0) = 1 \right\}$$

   Find the extrema $\bar{y}(x)$ of J over $\mathcal{A}$.

4. Consider the Fredholm integral equation

$$Ku = \lambda u(x) + f(x)$$

where the integral operator

$$Ku \equiv \int_0^1 k(x,y)u(y)\ dy$$

has the degenerate kernel

$$k(x,y) = 12y^3 - 30x^2y^2$$

Solve the equation for $\lambda = 1$ and $f(x) = 56x^2$.

5. Find the Green's function $G(x,\zeta)$ satisfying

$$-\frac{d}{dx}\left(x^2\frac{dG}{dx}\right) = \delta(x-\zeta) \qquad , \qquad x \in [1,2]$$

and the boundary conditions $G(1,\zeta) = G(2,\zeta) = 0$.

6. Find the general solution of

$$u_{xt} + \frac{2}{x}\ u_t = t \qquad , \qquad x \in \mathbb{R} \qquad , \qquad t > 0$$

and then determine the everywhere nonsingular solution of:

$$\begin{aligned} u_{xt} + \frac{2}{x}\ u_t &= t \\ u(x,0) &= \sin(x) \qquad , \qquad x > 0 \end{aligned}$$