Dynamical Systems M.S. exam 2003

1. Find bifurcation value(s) of the parameter $a \in R$ and determine the type of bifurcation(s) for differential equation on a circle ($\theta \in S^1$)

$$\dot{\theta} = 1 - a\sin\theta.$$

2. Consider a second order differential equation

$$m\ddot{x} = F(x) \tag{1}$$

where mass $m = 1$ and the force has the form $F(x) = x - x^2$.

a. Write the equation as a system and verify that the function

$$E(x, \dot{x}) = \frac{x^2 - \dot{x}^2}{2} + \frac{x^3}{3} + C,$$

where C is an arbitrary constant, is a conserved quantity (energy) for the system.

b. Draw a detailed phase portrait of the system; find all equilibria and their type and stability. Are there any periodic solutions for this problem?

c. Modify the equation (1) to equation

$$m\ddot{x} = F(x) - 0.01\dot{x}.$$

Write this equation as a system and discuss the behavior of energy $E(x, y)$ along the solutions of this system.

d. Draw a phase portrait of this system. Are there any periodic solutions for this problem? If so, find them, if not, argue why it is the case.

3. In the following questions you must decide whether the statement is true or false. In both cases you must support your statement by a picture, citation of a known result, or both.

a. There is no continuous map $f : [0, 1] \to [0, 1]$ which has fixed points, but no periodic points with minimal period greater then one.

b. There is a continuous map $f : [0, 1] \to [0, 1]$ which has only period two points, but no fixed points.

c. There is a continuous map $f : [0, 1] \to [0, 1]$ which has only period two points and fixed points.

4. Below is a partial picture of a continuous map $g : [0, 1] \to [0, 1]$. The map evidently has a fixed point. Prove that no matter how the graph of the rest of the function looks like, it always has a periodic point of minimal period 5.