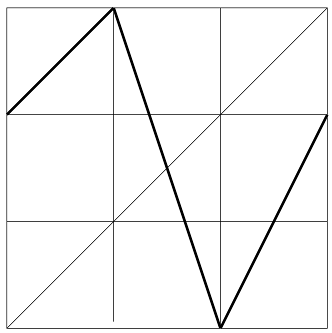
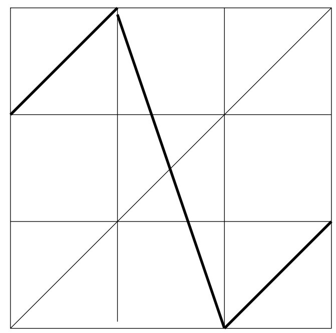
Dynamical Systems M.S. exam 2008

1. Consider maps $f, g : [0,1] \to [0,1]$ depicted below.



f



g

a) Show that one of the maps has periodic points of all periods.
b) Show that the other map has only points of period two and a unique fixed point. Find all of them.

2. Consider the following system of differential equations in R^2

$$\begin{aligned} \dot{x} &= y(y+x-2) && (1) \\ \dot{y} &= y+(x-2)^2-2. && (2) \end{aligned}$$

a) Find all the nullclines and draw a phase portrait.
b) Find and using linearization, classify all the equilibrium points.

3. A *hypercycle system* of the length 3 is a system of differential equation of the form

$$\begin{aligned} \dot{x}_1 &= -x_1 \pm a f_1(x_3) \\ \dot{x}_2 &= -x_2 + f_2(x_1) \\ \dot{x}_3 &= -x_3 + f_3(x_2) \end{aligned}$$

where , $a \geq 0$, all functions $f_k(\cdot), k = 1,2,3$ are monotonically increasing and satisfy

$$f_k(0) = 0 \text{ and } f'_k(0) = 1 \text{ for all } k.$$

If the sign in the first equation is $''+''$ this is called a *positive feedback system* and if it is $''-''$ the system is a *negative feedback system*.

a) Show that $\mathbf{0} := (0, 0, 0)$ is an equilibrium of the system.
b) With a as a paramter, show that for negative feeback system the equilibrium $\mathbf{0}$ looses stability via a Hopf bifurcation. Compute the value of $a > 0$ at the bifurcation.
c) Show that for a positive feedback system the equilibrium $\mathbf{0}$ looses stability via a pitchfork bifurcation. Compute the value of a at the bifurcation.

Hint: You will use your knowledge of finding roots of $z^3 = \pm A$.