M.S. Dynamical Systems Exam 2009

(Department of Mathematical Sciences, M.S.U.)

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Instructions: Attempt all questions. Show all work.

1. Define the planar system:

$$\begin{aligned} \dot{x} &= -\mu x + y \quad , \quad \mu \in \mathbb{R} \\ \dot{y} &= y - x^3 \end{aligned}$$

a) Find all the equilibria and their stability for all $\mu \in \mathbb{R}$.

b) Draw a bifurcation diagram of the system's equilibria in the (μ, x)-plane.

c) Does the system have any periodic orbits for $\mu > 0$? Explain.

2. In polar coordinates (r, θ) planar flow is described by:

$$\frac{dr}{dt} = g(r) = r\,(\,\mu - \,\, f(r)\,) \quad , \quad \mu \in \mathbb{R} \tag{1}$$

$$\frac{d\theta}{dt} = \frac{f(r)}{\cos\theta + 2} \tag{2}$$

where $f(r)$ is a smooth function.

a) Let $f(r) = r^2 - 2r + 2$. Determine the radius and stability of all periodic orbits when $\mu = \frac{5}{4}$.

b) Suppose that all one knows about the function $f(r)$ is that $\mu - f(r) = 0$ has a single root positive root r for all μ. Does the period of these orbits increase or decrease with μ? Hint: Integrate one of the equations.

c) Suppose all one knows about the function $f(r)$ is that $\mu - f(0) > 0$. For this case, explain why the system is or is not Hamiltonan. Hint: $g'(0)$ and equilibria stability

3. Define the map $x \mapsto f(x)$ as follows

$$f(x) = \begin{cases} ax & , \; x \leq 1 \\ a(2-x) & , \; x \geq 1 \end{cases}$$

where $a > 0$.

a) For what $a > 0$ does f map $[0, 2]$ into $[0, 2]$?

b) For what $a > 0$ does f have two fixed points?

c) For $a \in (1, 2)$ draw a qualitatively accurate figure of the second iterate map $f^2(x)$.

d) For what $a \in (0, 2)$ does f have minimal period 2 orbits?