Dynamical Systems M.S. exam 2010

Provide solid mathematical reasons for all your answers.

1. Consider the system of equations

$$\begin{aligned}\dot{x} &= x(2 - x - 2y)\\ \dot{y} &= y(2 - y - 2x).\end{aligned}$$

in the positive quadrant $Q := \{(x, y) : \ x, y \geq 0\}$ of R^2. Answer the following questions and then draw the phase portrait of the system.

a. Find all the equilibrium points and determine their stability.

b. Compute the linear approximation of the stable and unstable manifolds of the equilibria.

c. Find all the lines of the form $ax + by = 0$ that are invariant under the dynamics.

d. Show that there are no periodic orbits in Q.

e. Show that there is $l > 0$ such that $R := [0, l] \times [0, l]$ is positively invariant (i.e. if an initial condition is in R then so is the whole solution curve for $t \geq 0$).

f. Use part **e.** to describe basins of attraction of all invariant sets in Q.

2. Let $f(x) = 5x(1 - x)$ and $\Lambda \subset [0, 1]$ be the usual f-invariant Cantor set.

a. Let $Per_f := \{x \in \Lambda \mid x \text{ is periodic under } f\}$. Show that Per_f is dense in Λ (i.e. cl $Per_f = \Lambda$.)

b. Let $x_0 \in \Lambda$. Show that for all $\epsilon > 0$ there exists $x_\epsilon \in [0, 1]$ such that

1. $|x_0 - x_\epsilon| < \epsilon$;
2. x_ϵ has a dense orbit in Λ under f.

3. Let $L : R^3 \to R^3$ where

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

a. Find the stable manifold $W^s(0,0,0)$ and the unstable manifold $W^u(0,0,0)$ of the fixed point $(0,0,0)^T$.

b. Find all L-periodic points and their prime periods.

c. Show that L is not structurally stable.