M.S. Dynamical Systems Exam 2012

Instructions: Attempt all questions and Show all work. Good Luck!

Problem 1. Let $f(x) = 5x(1-x)$ and $\Lambda \subset [0,1]$ be the usual f-invariant Cantor set.

a. Describe an $x_0 \in [0,1]$ which has a dense orbit under f in Λ (i.e. the closure of the f-orbit of x_0 is Λ). Prove your choice of x_0 has the desired property.

b. Sketch the graph of $f : \mathbb{R} \to \mathbb{R}$. If $x_1 \notin \Lambda$, prove that

$$\lim_{n\to\infty} f^{(n)}(x_1) = -\infty.$$

Problem 2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the identity map (i.e. $f(p) = p$ for all $p \in \mathbb{R}^2$). Prove that for all $\epsilon > 0$ and all natural numbers n, that there exist a continuous map $g : \mathbb{R}^2 \to \mathbb{R}^2$ with the C^0 distance from f to g less then ϵ (i.e. $\sup_{p\in\mathbb{R}^2} ||f(p) - g(p)|| < \epsilon$), such that g has an orbit of prime period n.

Problem 3. Given the second order differential equation $x'' = x - x^2$.

a. Write the second order equation as a first order system.

b. Find all equilibrium points for your system.

c. Show how linearization predicts that one of these is a center and the other is a saddle.

d. Prove that the center is indeed an nonlinear center by finding a conserved quantity.

e. Given that the eigenvectors and eigenvalues for the saddle are $v_1 = (1,1)$ for $\lambda_1 = 1$ and $v_2 = (1,-1)$ for $\lambda_2 = -1$ and using the information above sketch a plausible phase portrait for the system.

Problem 4. An attractor is a set to which all neighboring trajectories converge. Asymptotically stable equilibrium points and stable limit cycles are examples. More precisely, we define an attractor to be a closed set A with the following properties:

i. A is an invariant set: any trajectory $\mathbf{x}(t)$ that starts in A stays in A for all time (forward and backward).

ii. A attracts an open set of initial conditions: there is an open set U containing A such that if $\mathbf{x}(0) \in U$ then the distance from $\mathbf{x}(t)$ to A tends to zero as $t \to \infty$. The largest such set U is called the basin of attraction of A.

iii. A is minimal: there is no proper subset of A that satisfies conditions i and ii.

Consider the following system in polar coordinates: $\dot{r} = r(1-r^2)$, $\dot{\theta} = 1$. Let D be the disk: $D := \{(x,y) | x^2 + y^2 \le 1\}$.

a. Is D an invariant set?

b. Does D attract an open set of initial conditions?

c. Is D an attractor? If not, why not? If so, find its basin of attraction.

d. Repeat part (c) above for the circle $S^1 = \{(x,y) | x^2 + y^2 = 1\}$.