Dynamical Systems M.S. exam 2013

Provide solid mathematical reasons for all your answers.

1. Consider the system

$$\dot{r} = r(1 - r^2) + \mu \cos(\theta), \qquad \dot{\theta} = 1.$$

Show that the system has a periodic orbit for $0 < \mu < 1$.

2. Classify bifurcations of the differential equation

$$\dot{x} = rx + 1 - \cos x$$

as a function of the parameter $r \in (-\infty, \infty)$. Whenever possible, find bifurcation values of parameters.

3. Consider the system of differential equations

$$\begin{aligned} \dot{x} &= -x - 2y^2 \\ \dot{y} &= xy - x^2 y \end{aligned} \qquad (1)$$

in the (x, y)- plane.

(a) Determine the nullclines of the system (1) and show that it has a single equilibrium (x^*, y^*) .

(b) Show that

$$V(x, y) = (x - x^*)^2 + a(y - y^*)^2$$

is a Lyapunov function for a suitably chosen value of a, and discuss the stability of the equilibrium.

(c) Sketch the phase portrait of the system, clearly indicating nullclines and the direction of the flow in the regions separated by nullclines. Sketch a typical trajectory starting from a point $x_0 > 1, y_0 > 0$.

4. Consider a map $f : [0, 1] \to [0, 1]$. A point p is *non-wandering*, denoted $p \in \Omega(f)$, if for any open interval J containing p, there exists $x \in J$ and $n > 0$ such that $f^n(x) \in J$.

(a) Prove that $\Omega(f)$ is a closed set.

(b) Let $F_\lambda(x) = \lambda x(1-x), F_\lambda : [0,1] \to [0,1]$. Assume $\lambda > 2+\sqrt{5}$ and denote by Λ the maximal invariant set of F_λ. Show this Cantor set Λ is equal to the non-wandering set $\Omega(F_\lambda)$.

5. A homeomorphism $f : R^2 \to R^2$ is *flowable* if there exists a flow φ_t on R^2 such that f is the time-one map of the flow φ_t, that is $f = \varphi_1$.

If a homeomorphism $g : R^2 \to R^2$ contains a horseshoe, show that g is not flowable.