Dynamical Systems M.S. exam 2014

Provide solid mathematical reasons for all your answers.

1.

(a.) Consider the system of differential equations

$$\begin{aligned} \dot{x} &= y - y^2 \\ \dot{y} &= 3 + x. \end{aligned}$$

(b.) Find all equilibria and classify them (stable node, unstable node, saddle, center, spiral sink, spiral source,...)

(c.) Find conserved quantity.

(d.) Using the conserved quantity to draw a detailed phase portrait.

2. Suppose that the overdamped pendulum is connected to a torsional spring. As the pendulum rotates, the spring winds up and generates an opposing torque $-k\theta$. Then the equation of motion becomes

$$b\dot{\theta} + mgL\sin\theta = \Gamma - k\theta.$$

(a.) Does this equation give a well defined vector field on the circle?

(b.) Non-dimensionalize the equation.

(c.) What does the pendulum do in the long run?

(d.) Show that many bifurcations occur as k is varied from 0 to ∞. What kind of bifurcations are they?

3. Consider a non-autonomous equation

$$\dot{x} = x(1 - 2xsin(t)). \tag{1}$$

(a.) Define a Poincaré map on an interval $[a, b]$ by following solutions for a time $t = 2\pi$, where $a < b$ are suitably chosen numbers.

(b.) Find a and b in such a way that there is a periodic solution $x_0(t)$ of (1) such that

$$a < x_0(t) < b \text{ for all } t.$$

4. Consider the area preserving baker's map $B : [0,1] \times\ [0,1] \to [0,1] \times [0,1]$, given by

$$(x_{n+}, y_{n+1}) = \begin{cases} (2x_n, \frac{1}{2}y_n) & \text{for } 0 \leq x_n < \frac{1}{2} \\ (2x_n - 1, \frac{1}{2}y_n + \frac{1}{2}) & \text{for } \frac{1}{2} \leq x_n \leq 1 \end{cases}.$$

(a.) Given that $(.a_1a_2a_3\ldots, .b_1b_2b_3\ldots)$ is the binary representation of an arbitrary point in a square $[0,1] \times [0,1]$, write down the binary representation of $B(x,y)$.

(b.) Using part (a) (or otherwise) show that B has a period-2 orbit, and sketch its location in the unit square.

(c.) Show that B has countably many periodic orbits.

(d.) Show that B has uncountably many aperiodic orbits.

(e.) Are there any dense orbits? If so, write down one explicitly. If not, explain why not.