Linear Algebra Masters Comprehensive Exam August 2015
Assume that all Vector Spaces are Finite Dimensional
Do Four of the following Six Problems and show all work.

1. Define $T \in \mathcal{L}(\mathcal{P}_2(\mathbf{R}))$ by $T(f) = x\frac{df}{dx}$, so for example $T(x^2) = x(2x) = 2x^2$.

   (a) Find the matrix of T with respect to the basis $(1, x, x^2)$.

   (b) Find T'.

   (c) Find $N_{T'}$.

   (d) Now make $\mathcal{P}_2(\mathbf{R})$ into an inner-product space by defining

   $$\langle p, q \rangle = \int_0^1 p(x)q(x)\,dx.$$

   Is T self-adjoint?

2. Let

   $$A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 2 & 6 \end{bmatrix}.$$

   (a) Find N_{A^T}.

   (b) Find conditions on b the guarantee that $Ax = b$ has a solution.

3. Prove that if $T \in \mathcal{L}(V)$ is normal, then

   $$\operatorname{range} T = \operatorname{range} T^*.$$

4. Prove that any linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

5. Suppose that $T \in \mathcal{L}(V, W)$ is injective (1-to-1) and $(v_1, \ldots, v_n)$ is linearly independent in V. Prove that $(Tv_1, \ldots, Tv_n)$ is linearly independent in W.

6. Prove that if $T \in \mathcal{L}(V)$ is self-adjoint, then the singular values of T equal the absolute values of the eigenvalues of T (repeated appropriately).