**Linear Algebra Masters Comprehensive Exam August 2017**
**Assume that all Vector Spaces are Finite Dimensional**
**Do five of the following seven Problems and show all work.**

1. Define $T \in \mathcal{L}(\mathcal{P}_2(\mathbf{R}))$ by $T(p) = x\frac{dp}{dx}$, so for example $T(x^2) = x(2x) = 2x^2$.

   (a) Find the matrix of $T$ with respect to the basis $(1, x, x^2)$.

   (b) Find $T'$ where $T'$ denotes the duel map for $T$.

   (c) Find $N_{T'}$.

   (d) Now make $\mathcal{P}_2(\mathbf{R})$ into an inner-product space by defining

   $$\langle p, q \rangle = \int_0^1 p(x)q(x)\, dx.$$

   Is $T$ self-adjoint?

2. 

   (a) Suppose $T \in \mathcal{L}(V)$ is diagonalizable. Prove that $V = \operatorname{null} T \oplus \operatorname{range} T$.

   (b) Prove the converse of the statement in the part (a) above or give a counterexample to the converse.

3. Let

   $$A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 2 & 6 \end{bmatrix}.$$

   (a) Find $N_{A^T}$.

   (b) Find conditions on $b$ the guarantee that $Ax = b$ has a solution.

3. Let $T$ be self-adjoint. Show that

   $$\operatorname{null} T = (\operatorname{range} T)^{\perp}$$

4. Suppose $V$ and $W$ are both finite-dimensional. Prove that there exists an injective linear map from $V$ to $W$ if and only if $\dim V \leq \dim W$.

5.

(a) State the Cayley-Hamilton Theorem being sure to define all terms like the characteristic polynomial.

(b) Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbf{F})$ such that $T^{-1} = p(T)$. Hint: consider the minimal polynomial of $T$.

6. Suppose $T \in \mathcal{L}(V)$ and $s$ is a singular value of $T$. Prove that there exists a vector $v \in V$ such that $\|v\| = 1$ and $\|Tv\| = s$.

7. Suppose $V$ is finite-dimensional and $U$ and $W$ are subspaces of $V$ with $W^0 \subset U^0$. Prove that $U \subset W$.