LINEAR ALGEBRA EXAM-SOLS

August 2018

Instructions: Do as many problems as you can. Show all work. Carefully read and follow the directions. Clearly label your work and attach it to this sheet.

1. Let $T : V \to W$ be a surjective linear transformation of finite dimensional vector spaces over a field F. Show that there is a linear transformation $S : W \to V$ such that $T \circ S$ is the identity map on W.

SOLUTION: Since T is surjective we know $W = \operatorname{range} T$. Let $w_1, \ldots, w_m$ be a basis of W. Because T is surjective, for each j there exists $v_j \in V$ such that $w_j = Tv_j$. Define $S \in \mathcal{L}(W, V)$ by

$$S(a_1 w_1 + \cdots + a_m w_m) = a_1 v_1 + \cdots + a_m v_m.$$

Then

$$\begin{aligned}(TS)(a_1 w_1 + \cdots + a_m w_m) &= T(a_1 v_1 + \cdots + a_m v_m)\\ &= a_1 T v_1 + \cdots + a_m T v_m\\ &= a_1 w_1 + \cdots + a_m w_m.\end{aligned}$$

Thus TS is the identity map on W.

2. Define T be a linear operator $T : \mathcal{P}_1(\mathbf{R}) \to \mathcal{P}_2(\mathbf{R})$ given by $T(p)(x) = p'(x) + xp(x)$. Here $\mathcal{P}_i(\mathbf{R})$ is a set of polynomials of maximal degree less or equal to i.

(a) Find the matrix of T with respect to the basis $(1, x)$ for $\mathcal{P}_1(\mathbf{R})$ and $(1, x, x^2)$ for $\mathcal{P}_2(\mathbf{R})$, i.e., $M(T, (1, x), (1, x, x^2))$.

(b) Find T', the transpose operator.

(c) Find $N_{T'}$.

SOLUTION: a) Since $T(1) = x$ and $T(x) = 1 + x^2$,

$M(T,(1,x),(1,x,x^2)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$

b) Note, $T'(\varphi)$ for $\varphi \in \mathcal{P}_2'(\mathbf{R})$ is defined to be $\varphi \circ T$, i.e,

$$T'(\varphi)(p) := \varphi \circ T(p) \qquad \text{for } \varphi \in \mathcal{P}_2'(\mathbf{R})$$

and $p \in \mathcal{P}_1(\mathbf{R})$.

Let the dual basis of the basis for $1, x, x^2$ of $\mathcal{P}_2(\mathbf{R})$ be $\varphi_0, \varphi_1 \varphi_2$. Note that $\varphi_i(p_j) = \delta_{ij}$ so one checks that $\varphi_j(p) = \frac{p^{(j)}(0)}{j!}$. Here $p^{(j)}$ denotes the j^{th} derivative of p, with the understanding that the 0^{th} derivative of p is p. Likewise let the dual basis for $1, x$ of $\mathcal{P}_1(\mathbf{R})$ be be ψ_0, ψ_1, where as above we see that $\psi_j(p) = \frac{p^{(j)}(0)}{j!}$, for $p \in \mathcal{P}_1(\mathbf{R})$. If $\varphi \in \mathcal{P}_2'(\mathbf{R})$ then $\varphi = c_0\varphi_0 + c_1\varphi_1 + c_2\varphi_2$ since $\varphi_0, \varphi_1, \varphi_2$ is a basis for $\mathcal{P}_2'(\mathbf{R})$.

Note for any $p \in \mathcal{P}_1(\mathbf{R})$, we have $p(x) = \psi_0(p) + \psi_1(p)x$ so

$$\begin{aligned} T'(\varphi)(p) &= \varphi(T(\psi_0(p) + \psi_1(p)x) \\ &= \varphi(\psi_1(p) + \psi_0(p)x + \psi_1(p)x^2) \\ &= c_0\psi_1(p) + c_1\psi_0(p) + c_2\psi_1(p) \\ &= c_1\psi_0(p) + (c_0 + c_2)\psi_1(p). \end{aligned}$$

Thus we see that

$$T'(c_0\varphi_0 + c_1\varphi_1 + c_2\varphi_2) = c_1\psi_0 + (c_0 + c_2)\psi_1$$

c) From part b) above we see that $\varphi \in \operatorname{null}(T')$ if and only if $c_1 = 0$ and $c_0 = -c_2$ thus

$$\operatorname{null}(T') = \operatorname{span}(\varphi_2 - \varphi_0).$$

3. Let V be finite dimensional real inner product space and $L : V \to V$ be a self-adjoint isometry. Show that there exists a subspace $U \subset V$ such that

$$L(u+v) = u - v \quad \text{for all } u \in U \text{ and } v \in V.$$

SOLUTION: Clearly this is not correct as stated, for a counterexample take $L = I$ the identity map on V. This is a self-adjoint isometry and $I(u+v) \equiv u+v$ which is not $u-v$ for all $v \in V$.

The correct statement should have been:

Show that there exits a subspace $U \subset V$ such that

$$L(u+v) = u - v \quad \textit{for all } u \in U \textit{ and } v \in U^{\perp}.$$

To show this note that since L is self-adjoint all eigenvalues are real and by the Spectral Theorem, V has a basis of eigenvectors for L. Furthermore, eigenvectors corresponding to distinct eigenvalues are orthogonal. Since L is an isometry, the eigenvalues of L have complex modulus 1. Putting these two together, the only possible eigenvalues of L are ± 1.

Now let $U = \text{null}(L - I)$, then $V = U \oplus U^{\perp}$ and $U^{\perp} = \text{null}(L + I)$ since V has an orthonormal basis consisting of eigenvectors of L and the only eigenvalues of L can be ± 1.

Then if $u \in U$ and $v \in U^{\perp}$ then

$$L(u+v) = Lu + Lv = u - v.$$

4. Let V be finite dimensional complex vector space with norm $||\cdot||$ and $T : V \to V$ a linear operator. Assume that for every $\epsilon > 0$ there is $k \geq 1$ such that $||T^k u|| \leq \epsilon^k$ for all $u \in V$. Prove that $T^{\dim V} = 0$.

SOLUTION: Let $n := \dim V$. Then T being an operator on a complex vector space has n eigenvalues. Let (λ, v) be an arbitrary eigenpair with $||v|| = 1$. Then or any ϵ, there is k such that

$$||T^k v|| = |\lambda|^k ||v|| = |\lambda|^k \leq \epsilon^k,$$

which implies $|\lambda| \leq \epsilon$. We conclude that for any $\epsilon > 0$ we have $|\lambda| \leq \epsilon$. Therefore we must have $\lambda = 0$ and since λ was arbitrary, it must be that all eigenvalues of T are zero.

Therefore T is nillpotent with characteristic polynomial λ^n and by Cayley-Hamilton Theorem

$$T^n = T^{\dim V} = 0.$$

5. Let $L : V \to V$ be a linear transformation on a finite dimensional inner product space V. Show that if

$$||Lv|| = ||v|| \quad \text{for all } v \in V$$

then $LL^* = I$, the identity transformation on V.

SOLUTION: The assumption implies that $||Lv||^2 = ||v||^2$ which in turns implies that for all $v \in V$

$$\begin{aligned} \langle Lv, Lv\rangle &= \langle v, v\rangle \\ \langle L^*Lv, v\rangle &= \langle v, v\rangle \\ \langle (L^*L - I)v, v\rangle &= 0 \end{aligned}$$

Note that the oparator $T := L^*L - I$ is self-adjoint by direct inspection. Then By Proposition 7.16 if T is self-adjoint, $\langle Tv, v\rangle = 0$ for all $v \in V$ implies that $T = 0$. As a consequence $L^*L = I$ which implies that $L^{-1} = L^*$ and so $LL^* = I$ also.

6. Let A be 6×6 matrix with characteristic polynomial $(x-1)^4(x+2)^2$ and the minimal polynomial $(x-1)^2(x+2)^2$. What is the Jordan canonical form of A if the rank of $I - A$ is 3?

SOLUTION: From characteristic polynomial set of eigenvalues is $\{1, 1, 1, 1, -2, -2\}$. Since the minimal polynomial has a term $(x+2)^2$, there must be a 2×2 Jordan block with -2 on a diagonal and 1 above the diagonal.

By similar reason, the Jordan block corresponding to eigenvalue 1 can only consist of sub-blocks of size 1 or 2, and there must be at least one block of size 2. Otherwise, if there is a block of size 4 then minimal polynomial would have factor $(x + 1)^4$, if there was a block of size 3, it would have a factor $(x+1)^3$ and if there was no block of size 2 then the minimal polynomial would have a factor $x + 1$.

So we are left with two possibilities: either there is a single 2×2 block and two blocks of size 1, or there are two blocks of size 2, corresponding to eigenvalue 1. By direct observation, in the first case $rank(I - A) = 3$ (two from the 2×2 block corresponding to -2 and one from the single 2×2 block corresponding to eigenvalue 1; in the second case $rank(I - A) = 4$ (two from the 2×2 block corresponding to -2 and two from the single 2×2 block corresponding to eigenvalue 1. Therefore the Jordan form is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}.$$

Obviously, permutations of the first four basis elements are permissible and will produce what is considered to be the same Jordan form.