MS Exam: LINEAR ALGEBRA

Instructions: Attempt all problems, showing work. Note: all vector spaces are finite dimensional.

1. For a linear transformation $T : \mathbf{V} \to \mathbf{V}$, where $\mathbf{V}$ is a real vector space of dimension n, $0 < n < \infty$, let $\mathbf{V}_0$ denote the range of T and $T_0 : \mathbf{V}_0 \to \mathbf{V}_0$ the restriction of T to $\mathbf{V}_0$.

(a) Is T_0 an isomorphism? Justify your answer.

(b) If T_0 is an isomorphism, and the *minimal polynomial* of T is divisible by λ^k then show that $k = 0$ or $k = 1$.

2. Let $(\mathbf{V}, \langle \cdot, \cdot \rangle)$ denote a complex inner product space. Suppose the operator $T : \mathbf{V} \to \mathbf{V}$ is normal, with $T \neq 0$, and also has the property that $P = T^*T$ is an *orthogonal projection* on $\mathbf{V}$. Show that there exists a *nonzero* subspace $\mathbf{W} \subset \mathbf{V}$, such that $||Tw|| = ||w||$ for each $w \in \mathbf{V}$. (Use the spectral theorem.)

3. For parts (a) and (c) you may answer without proof.
(a) Find the jordan form of the following matrix (the field is $\mathbf{C}$).

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

(b) Find the jordan form of the following matrix (the field is $\mathbf{C}$).

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

(c) Find all possible jordan forms for a (3×3)-nilpotent matrix N. Recall that N is nilpotent if $N^k = 0$ for some positive integer k.