MS Exam: LINEAR ALGEBRA

Instructions: Attempt all problems, showing work. Note: all vector spaces are finite dimensional.

- 1. For a linear transformation $T: \mathbf{V} \to \mathbf{V}$, where \mathbf{V} is a real vector space of dimension $n, 0 < n < \infty$, let \mathbf{V}_0 denote the range of T and $T_0: \mathbf{V}_0 \to \mathbf{V}_0$ the restriction of T to \mathbf{V}_0 .
- (a) Is T_0 an isomorphism? Justify your answer.
- (b) If T_0 is an isomorphism, and the *minimal polynomial* of T is divisible by λ^k then show that k = 0 or k = 1.
- 2. Let $(\mathbf{V}, \langle \cdot, \cdot \rangle)$ denote a complex inner product space. Suppose the operator $T: \mathbf{V} \to \mathbf{V}$ is normal, with $T \neq 0$, and also has the property that $P = T^*T$ is an orthogonal projection on \mathbf{V} . Show that there exists a nonzero subspace $\mathbf{W} \subset \mathbf{V}$, such that ||Tw|| = ||w|| for each $w \in \mathbf{V}$. (Use the spectral theorem.)
- 3. For parts (a) and (c) you may answer without proof.
- (a) Find the jordan form of the following matrix (the field is C).

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

(b) Find the jordan form of the following matrix (the field is C).

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

(c) Find all possible jordan forms for a (3×3) -nilpotent matrix N. Recall that N is nilpotent if $N^k=0$ for some positive integer k.