M.S. COMPREHENSIVE EXAMINATION IN LINEAR ALGEBRA
January 14, 2002

Do as much as you can on each problem. If you cannot do a part, assume that result and go on to the next part.

1. Let the matrix A be given by

$$A = \begin{bmatrix} 0 & -2 & 5 \\ 0 & 1 & 4 \\ 0 & -1 & 5 \end{bmatrix} .$$

(a) Show that the matrix A has eigenvalues $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 3$ and corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} .$$

(b) What is $\det(A)$? ______ Is A singular or nonsingular? ____________

(c) Give a basis for the eigenspace of A corresponding to the eigenvalue $\lambda_2 = \lambda_3 = 3$.

(d) Give a basis for the null space of A.

(e) Give a basis for the column space of A.

(f) If A is diagonalizable, identify the matrices P and Λ such that $P^{-1}AP = \Lambda$. If A is not diagonalizable, explain why not.

2. Let A be an $n \times n$ real symmetric matrix. Let the eigenpairs of A be $(\lambda_1, \mathbf{v}_1)$, $(\lambda_2, \mathbf{v}_2)$, ... , $(\lambda_n, \mathbf{v}_n)$, where the eigenvalues are distinct and the eigenvectors are chosen to be orthonormal.

(a) Using the vectors $\mathbf{v}_i$ defined above, carefully show that for any $n \times n$ matrix B, if $B\mathbf{v}_i = 0$ for all $i = 1, 2, \ldots, n$, then $B \equiv \mathcal{O}$, the zero matrix.

(b) Use part (a) to show that

$$A = \lambda_1 \mathbf{v}_1 \mathbf{v}_1^T + \lambda_2 \mathbf{v}_2 \mathbf{v}_2^T + \cdots + \lambda_n \mathbf{v}_n \mathbf{v}_n^T .$$

3. Let the $n \times n$ matrices A and B be diagonalizable with distinct eigenvalues.

(a) Prove that if A and B can be simultaneously diagonalized (diagonalized by the same matrix P so that $P^{-1}AP = \Lambda_1$ and $P^{-1}BP = \Lambda_2$), then A and B commute ($AB = BA$).

(b) Prove that if A and B commute then A and B can be simultaneously diagonalized. Start by letting P be the diagonalizer of A and hence $P^{-1}AP = \Lambda_1$. Then denote $P^{-1}BP$ by C ($P^{-1}BP = C$). Show that Λ_1 and C commute. Now show C is diagonal.