Linear Algebra M.S. Comprehensive Examination

January 2008

Name:

1. Let V be a space of all real 2×2 matrices. Consider transformation $\mathcal{A} : V \to V$ given by

$$\mathcal{A}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \to \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix}.$$

a) Show that $\mathcal{A}$ is a linear transformation.
b) Find a basis of V and find the matrix of $\mathcal{A}$ in this basis.
c) Find the nullspace of $\mathcal{A}$ and its dimension.

2. Suppose an inner product space X admits an orthogonal decomposition $X = U \oplus V \oplus W$. Let P_U be the orthogonal projection onto the subspace U, P_V be the orthogonal projection onto the subspace V and P_W be the orthogonal projection onto the subspace W.

a. Find the nullspace of $P_U + P_W$ and $P_U - P_V$.

b. Do P_U and P_V commute? Prove, or provide a counterexample.

c. Does b) hold if we replace $X = U \oplus V \oplus W$ by $X = U + V + W$, but keep the assumption that P_U, P_V and P_W are orthogonal projections? Prove, or find a counterexample.

3. Prove or find a counterexample.

a. A sum of two positive operators is a positive operator.

b. A sum of two isometries is an isometry.

c. Every matrix can be written as a product of an orthogonal matrix and an upper-triangular marix. (A matrix is orthogonal if its columns are orthonormal.)

d. Every non-zero symmetric operator is invertible.

4. Consider a real 6×6 matrix A and assume that

1. 3 is an eigenavalue of A;

2. there mutually perpendicular vectors u, v, w such that

$$\begin{aligned} (A-I)^2 u = 0, &\qquad (A-I)u \neq 0 \\ (A-2I)^2 v = 0, &\qquad (A-2I)v \neq 0 \\ (A-2I)w = 0 &\qquad . \end{aligned}$$

a. Find the characteristic polynomial of A.

b. Find the minimal polynomial of A.

c. Find the Jordan normal form of A.