LINEAR ALGEBRA MASTER'S EXAM

January 2012

Instructions: Solve 4 of the following 5 questions. Show all work. Carefully read and follow the directions. Clearly label your work and attach it to this sheet.

1. Let V be a finite dimensional inner product space with inner product denoted by $\langle \cdot, \cdot \rangle$, and suppose ϕ is a linear functional on V. Show that there exists a unique vector $v \in V$ such that

$$\phi(u) = \langle u, v \rangle$$

for every $u \in V$.

2. Suppose V is a finite dimensional inner product space, $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T if and only if $U^{\perp}$ is invariant under $T^{\star}$.

3. Suppose P is an orthogonal projection onto a subspace S and Q is an orthogonal projection onto the orthogonal complement $S^{\perp}$.

   (a) Describe the operators $P + Q$ and PQ.

   (b) Show that $P - Q$ is its own inverse.

4. Define $V = M_{22}(\mathbb{R})$ and $T \in \mathcal{L}(V)$ by

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

Determine whether T is normal, self-adjoint or neither. And if possible, produce an orthonormal basis of eigenvectors of T for V. List the corresponding eigenvalues.

5. Find $p \in P_3(\mathbb{R})$ such that $p(0) = 0$, $p'(0) = 0$ and

$$\int_0^1 |2 + 3x - p(x)|^2 dx$$

is as small as possible.