LINEAR ALGEBRA MASTER'S EXAM

January 2015

Instructions: Solve all 4 of the following problems. Show all work. Carefully read and follow the directions. Clearly label your work and attach it to this sheet.

1. Prove that if W_1 and W_2 are subspaces of a finite dimensional vector space, then the subspace $W_1 + W_2$ is finite dimensional, and

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

2. Let $V = P_3(\mathbb{R})$, and define $T \in \mathcal{L}(V)$ by

$$T(a + bx + cx^2 + dx^3) = -d + (-c + d)x + (a + b - 2c)x^2 + (-b + c - 2d)x^3$$

   Let $B = (1 - x + x^3, 1 + x^2, 1, x + x^2)$ denote a basis of V.

   (a) Compute $\mathcal{M}(T)$, the matrix of T, with respect to the basis B.

   (b) Determine whether or not B is a basis consisting of eigenvectors of T.

3. Suppose $T \in \mathcal{L}(V, W)$, and let T^* denote the adjoint of T.

   (a) Prove that $\dim(\text{range}(T)) = \dim(\text{range}(T^*))$.

   (b) Prove that T is injective if and only if T^* is surjective.

   (c) Let A be a 3×5 matrix with real entries, and assume that A has three linearly independent columns. Determine the dimension of the null space $\dim(\text{Null}(A))$.

4. For the following matrix A

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ \sqrt{3} & 0 \end{bmatrix}$$

   (a) Give the Singular Value Decomposition of A.

   (b) Let $U = \text{span}(\mathbf{u}_1)$, where $\mathbf{u}_1$ is the first principal component of A; that is, $\mathbf{u}_1$ is the left singular vector of A corresponding to the largest singular value of A. Let $P_U : R^3 \to R^3$ denote the projection operator that projects onto the subspace U. Construct the matrix $\mathcal{M}(P_U)$ with respect to the standard basis.