January 2009 Probability / Math Stat Comprehensive Exam (100 Points)

1. Suppose $P(A) = P(B) = 1/3$ and $P(A \cap B) = 1/10$. Find:

   (a) (2pt) $P(A \cup B)$

   (b) (3pt) $P(B|A)$

   (c) (3pt) $P(A \cup B')$

2. X is a continuous random variable with pdf $f(x) = \begin{cases} 3/x^4 & \text{if } 1 < x < \infty \\ 0 & \text{otherwise} \end{cases}$

   (a) (4pt) Find $E(X)$.

   (b) (4pt) Find the CDF $F(x)$. Be sure to define $F(x)$ for all real numbers.

3. (9pt) The probability that a firecracker successfully explodes when ignited is 0.9. Consider the following random variables (X, Y, and Z).

   - Let X be the number of firecrackers tested when the first firecracker tested successfully explodes.
   - Let Y be the number of firecrackers tested when the third firecracker tested successfully explodes.
   - Let Z be the number of firecrackers that successfully explode in a random sample of 10 firecrackers.

   State the distributions of X, Y, and Z. Be sure to specify the associated parameter values.

4. (4pt) X and Y are random variables with joint pdf $f(x, y) = \begin{cases} 8xy & \text{for } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$

   (a) (2pt) Sketch and shade in the region of support for the joint pdf.

   (b) (2pt) Find $E(XY)$.

   (c) (2pt) Find $f_X(x)$, the marginal pdf for X.

   (d) (2pt) Find $f_{Y|X}(y|x)$, the conditional pdf for $Y|X$.

5. Two discrete random variables X and Y have joint pdf defined by the following table:

		Y = 1	Y = 2
X	1	.20	.60
	2	.05	.15

   (a) (2pt) What is the marginal pdf $f_Y(y)$?

   (b) (2pt) What is the conditional pdf $f(Y|X = 1)$?

   (c) (2pt) Are X and Y independent? Justify your answer.

6. (6pt) Find the pdf of $Y = X^4$ if X is a random variable with pdf

$$f_X(x) = 4x^3 \quad \text{for } 0 < x < 1 \quad \text{(and is 0 otherwise)}.$$

7. Suppose $X_1 \sim \mathrm{EXP}(4)$, $X_2 \sim \mathrm{GAMMA}(4,4)$, and X_1 and X_2 are independent.

   (a) (3pt) What is the moment generating function of $Y = X_1 + X_2$?

   (b) (3pt) Identify the distribution of Y?

8. Let $X_1, X_2, \ldots, X_n$ be a random sample from a weibull $\mathrm{WEI}(\theta, 1/2)$ distribution.

   (a) (4pt) Find the method of moments estimator (MME) of θ.

   (b) (5pt) Verify the maximum likelihood estimator (MLE) of θ is $\widehat{\theta} = \left(\frac{\sum_{i=1}^{n}\sqrt{x_i}}{n}\right)^2$. (You do not need to verify that it is a maximum with a second derivative test).

   (c) (2pt) What is the MLE of $\sqrt{\theta}$?

9. Let $X_1, X_2, \ldots, X_n$ be a random sample from a geometric $\mathrm{GEO}(p)$ distribution.

   (a) (5pt) Determine the Cramer-Rao lower bound (CRLB) for $\overline{X}$ which is an unbiased estimator of $\tau(p) = \frac{1}{p}$. Hint: $\dfrac{\partial}{\partial p} \ln f(x;p) = \dfrac{x - \frac{1}{p}}{p-1}$.

   (b) (3pt) Verify that $\overline{X}$ is a UMVUE of $\frac{1}{p}$ using the CRLB in (a).

10. Let $X_1, ..., X_{25}$ be a random sample from an EXP(2).

    (a) (5pt) In this example we could use the Central Limit Theorem to approximate the distribution of $\overline{X}$. What is that approximate distribution of $\overline{X}$ based on the Central Limit Theorem? Be sure to state the values of the associated parameters.

    (b) (4pt) Suppose $X_1, \ldots, X_{25}$ is a random sample from an EXP(2) distribution. Approximate the probability that $\overline{X} \geq 2.5$.

11. (8pt) Suppose X and Y are independent EXP(1) random variables. Find the joint pdf of $S = X + Y$ and $T = X$.

12. (6pt) Find a complete sufficient statistic if we have a random sample $X_1, X_2, \ldots, X_n$ from a distribution with pdf $f(x;\theta) = \dfrac{(\ln\theta)\theta^x}{\theta - 1}$ for $0 < x < 1$ and $\theta > 1$.

13. (7pt) Let X_1, X_2, X_3, X_4, X_5 be a random sample of size $n = 5$ from a $\mathrm{Gamma}(2, 1/\theta)$ pdf. It can be shown that $2\theta X_i \sim \chi^2(4)$. Determine an equal-tailed 90% interval estimator for θ based on $Y = \sum_{i=1}^{5} 2\theta X_i$.

Special Discrete Distributions

Notation and Parameters	Discrete pdf $f(x)$	Mean	Variance	MGF $M_X(t)$
Binomial				
$X \sim \text{BIN}(n, p)$	$\binom{n}{x} p^x q^{n-x}$	np	npq	$(pe^t + q)^n$
$0 < p < 1$ $q = 1 - p$	$x = 0, 1, \ldots, n$			
Bernoulli				
$X \sim \text{BIN}(1, p)$	$p^x q^{1-x}$	p	pq	$pe^t + q$
$0 < p < 1$ $q = 1 - p$	$x = 0, 1$			
Negative Binomial				
$X \sim \text{NB}(r, p)$	$\binom{x-1}{r-1} p^r q^{x-r}$	r/p	rq/p^2	$\left(\frac{pe^t}{1 - qe^t}\right)^r$
$0 < p < 1$ $r = 1, 2, \ldots$	$x = r, r+1, \ldots$			
Geometric				
$X \sim \text{GEO}(p)$	pq^{x-1}	$1/p$	q/p^2	$\frac{pe^t}{1 - qe^t}$
$0 < p < 1$ $q = 1 - p$	$x = 1, 2, \ldots$			
Hypergeometric				
$X \sim \text{HYP}(n, M, N)$	$\binom{M}{x}\binom{N-M}{n-x} \Big/ \binom{N}{n}$	nM/N	$n\frac{M}{N}\left(1 - \frac{M}{N}\right)\frac{N-n}{N-1}$	*
$n = 1, 2, \ldots, N$ $M = 0, 1, \ldots, N$	$x = 0, 1, \ldots, n$			
Poisson				
$X \sim \text{POI}(\mu)$	$\frac{e^{-\mu}\mu^x}{x!}$	μ	μ	$e^{\mu(e^t - 1)}$
$0 < \mu$	$x = 0, 1, \ldots$			
Discrete Uniform				
$X \sim \text{DU}(N)$	$1/N$	$\frac{N+1}{2}$	$\frac{N^2 - 1}{12}$	$\frac{1}{N}\,\frac{e^t - e^{(N+1)t}}{1 - e^t}$
$N = 1, 2, \ldots$	$x = 1, 2, \ldots, N$			

*Not tractable.

Special Continuous Distributions

Notation and Parameters	Continuous pdf $f(x)$	Mean	Variance	MGF $M_X(t)$
Uniform				
$X \sim \text{UNIF}(a, b)$ $a < b$	$\frac{1}{b-a}$ $a < x < b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t}$
Normal				
$X \sim \text{N}(\mu, \sigma^2)$ $0 < \sigma^2$	$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-[(x-\mu)/\sigma]^2/2}$	μ	σ^2	$e^{\mu t + \sigma^2 t^2/2}$
Gamma				
$X \sim \text{GAM}(\theta, \kappa)$ $0 < \theta$ $0 < \kappa$	$\frac{1}{\theta^\kappa \Gamma(\kappa)} x^{\kappa-1} e^{-x/\theta}$ $0 < x$	$\kappa\theta$	$\kappa\theta^2$	$\left(\frac{1}{1-\theta t}\right)^\kappa$
Exponential				
$X \sim \text{EXP}(\theta)$ $0 < \theta$	$\frac{1}{\theta} e^{-x/\theta}$ $0 < x$	θ	θ^2	$\frac{1}{1-\theta t}$
Two-Parameter Exponential				
$X \sim \text{EXP}(\theta, \eta)$ $0 < \theta$	$\frac{1}{\theta} e^{-(x-\eta)/\theta}$ $\eta < x$	$\eta + \theta$	θ^2	$\frac{e^{\eta t}}{1-\theta t}$
Double Exponential				
$X \sim \text{DE}(\theta, \eta)$ $0 < \theta$	$\frac{1}{2\theta} e^{-\lvert x-\eta \rvert/\theta}$	η	$2\theta^2$	$\frac{e^{\eta t}}{1-\theta^2 t^2}$
Weibull				
$X \sim \text{WEI}(\theta, \beta)$ $0 < \theta$ $0 < \beta$	$\frac{\beta}{\theta^\beta} x^{\beta-1} e^{-(x/\theta)^\beta}$ $0 < x$	$\theta\Gamma\left(1+\frac{1}{\beta}\right)$	$\theta^2\left[\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma^2\left(1+\frac{1}{\beta}\right)\right]$	*
Extreme Value				
$X \sim \text{EV}(\theta, \eta)$ $0 < \theta$	$\frac{1}{\theta}\exp\{[(x-\eta)/\theta] - \exp[(x-\eta)/\theta]\}$	$\eta - \gamma\theta$ $\gamma \doteq 0.5772$ (Euler's const.)	$\frac{\pi^2\theta^2}{6}$	$e^{\eta t}\Gamma(1+\theta t)$

Special Continuous Distributions

Notation and Parameters	Continuous pdf $f(x)$	Mean	Variance	MGF $M_X(t)$
Cauchy				
$X \sim \text{CAU}(\theta, \eta)$	$\dfrac{1}{\theta\pi\{1 + [(x - \eta)/\theta]^2\}}$	**	**	**
$0 < \theta$				
Pareto				
$X \sim \text{PAR}(\theta, \kappa)$	$\dfrac{\kappa}{\theta(1 + x/\theta)^{\kappa+1}}$	$\dfrac{\theta}{\kappa - 1}$	$\dfrac{\theta^2\kappa}{(\kappa - 2)(\kappa - 1)^2}$	**
$0 < \theta$, $0 < \kappa$	$0 < x$	$1 < \kappa$	$2 < \kappa$	
Chi-Square				
$X \sim \chi^2(\nu)$	$\dfrac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2 - 1} e^{-x/2}$	ν	2ν	$\left(\dfrac{1}{1 - 2t}\right)^{\nu/2}$
$\nu = 1, 2, \ldots$	$0 < x$			

Notation and Parameters	Continuous pdf $f(x)$	Mean	Variance	MGF $M_X(t)$
Student's t				
$X \sim t(\nu)$	$\dfrac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \dfrac{1}{\sqrt{\nu\pi}} \left(1 + \dfrac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$	0	$\dfrac{\nu}{\nu - 2}$	**
$\nu = 1, 2, \ldots$		$1 < \nu$	$2 < \nu$	
Snedecor's F				
$X \sim \text{F}(\nu_1, \nu_2)$	$\dfrac{\Gamma\left(\frac{\nu_1 + \nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\dfrac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} x^{\frac{\nu_1}{2} - 1}$	$\dfrac{\nu_2}{\nu_2 - 2}$	$\dfrac{2\nu_2^2(\nu_1 + \nu_2 - 2)}{\nu_1(\nu_2 - 2)^2(\nu_2 - 4)}$	**
$\nu_1 = 1, 2, \ldots$, $\nu_2 = 1, 2, \ldots$	$\times \left(1 + \dfrac{\nu_1}{\nu_2} x\right)^{-\frac{\nu_1 + \nu_2}{2}}$	$2 < \nu_2$	$4 < \nu_2$	
Beta				
$X \sim \text{BETA}(a, b)$	$\dfrac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1 - x)^{b-1}$	$\dfrac{a}{a + b}$	$\dfrac{ab}{(a + b + 1)(a + b)^2}$	*
$0 < a$, $0 < b$	$0 < x < 1$			

*Not tractable.

**Does not exist.

Standard normal probabilities

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Standard normal probabilities

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Probability / Math Stat Handout to Accompany Exam

- If T is an unbiased estimator of $\tau(\theta)$, then the **Cramer-Rao lower bound (CRLB)** (given a random sample) is

$$\mathrm{Var}(T) \;\geq\; \frac{[\tau'(\theta)]^2}{n\mathrm{E}\left[\frac{\partial}{\partial\theta}\ln f(X;\theta)\right]^2}$$

assuming (i) the derivatives exist and (ii) they can be passed into the integral (continuous X) or summation (discrete X) when calculating expectations.

- Let statistic $S_i = \mathcal{S}_i(X_1,\ldots,X_n)$ and and $s_i = \mathcal{S}_i(x_1,\ldots,x_n)$ for $i=1,\ldots,k$. Let $\mathcal{S}(x_1,\ldots,x_n)$ be the vector of functions whose j^{th} coordinate is $\mathcal{S}_j(x_1,\ldots,x_n)$ Then, mathematically, if the conditional pdf of $\mathbf{X}$ given $\mathbf{S}=\mathbf{s}$,

$$f_{\mathbf{X}|\mathbf{s}}(x_1,x_2,\ldots,x_n) \;=\; \frac{f(x_1,x_2,\ldots,x_n;\theta)}{f_{\mathbf{S}}(\mathbf{s};\theta)} \quad \text{if } \mathcal{S}(x_1,\ldots,x_n)=\mathbf{s} \qquad \text{and } 0 \text{ otherwise}$$

does not depend on θ, then $\mathbf{S}$ is a set of sufficient statistics.

- **Theorem 10.2.1 (Factorization Criterion)**: If $X_1,\ldots,X_n$ is a random sample having joint pdf $f(x_1,\ldots,x_n;\theta)$, and if $\mathbf{S}=(S_1,\ldots,S_k)$, then $\mathbf{S}$ is jointly sufficient for θ if and only if

$$f(x_1,\ldots,x_n;\theta) \;=\; g(\mathbf{s};\theta)\,h(x_1,\ldots,x_n)$$

where $g(\mathbf{s};\theta)$ does not depend on observations $x_1,\ldots,x_n$ except through $\mathbf{s}$ and $h(x_1,\ldots,x_n)$ does not involve θ.

- **Theorem 10.3.1**: If $S_1,\ldots,S_k$ are jointly sufficient for θ and if $\widehat{\theta}$ is a unique MLE of θ, then $\widehat{\theta}$ is a function of $\mathbf{S}=(S_1,\ldots,S_k)$.

- A density function is a member of the **regular exponential class (REC)** if it can be expressed in the form

$$f(x;\theta) \;=\; c(\theta)h(x)\exp\left[\sum_{i=1}^{k} q_j(\theta)t_j(x)\right] \quad \text{for } x\in A \text{ (support)}$$

and is 0 otherwise, where $\theta=(\theta_1,\ldots,\theta_k)$ is a vector of unknown parameters and the parameter space Ω has the form

$$\Omega \;=\; \{\theta|\, a_i<\theta_i<b_i,\ i=1,\ldots,k\}$$

(Note: $a_i=-\infty$ and $b_i=\infty$ are allowable values.)

and regularity conditions are satisfied.

- **Theorem 10.4.2**: If $X_1,\ldots,X_n$ is a random sample from a member of the REC$(q_1,\ldots,q_k)$, then the statistics $S_1=\sum_{i=1}^{n}t_1(X_i)$, $\quad S_2=\sum_{i=1}^{n}t_2(X_i)$, $\quad\ldots,\quad S_k=\sum_{i=1}^{n}t_k(X_i)$ are a minimal set of complete sufficient statistics for $\theta_1,\ldots,\theta_k$.