

Probability Comprehensive Exam

Saturday, January 5, 2013

Show work for full credit. Start each problem on a new page.

1. (5 pts) Show that if A and B are mutually exclusive, then

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}$$

Assume that $P(A) > 0$ and $P(B) > 0$.

2. (5 pts) Prove that if $P(A) > 0$ and $P(B) > 0$ then
- (a) if A and B are mutually exclusive, they cannot be independent.
 - (b) if A and B are independent, they cannot be mutually exclusive.
3. (10 pts) When coded messages are sent, there are sometimes errors in transmission. In particular, Morse code uses "dots" and "dashes". For any given symbol,

$$P(\text{dot sent}) = 3/7 \quad \text{and} \quad P(\text{dash sent}) = 4/7$$

Suppose there is interference on the transmission line, and with probability $1/4$ a dot is mistakenly received as a dash and with probability $1/3$ a dash is mistakenly received as a dot. If a dash is received what is the probability that a dash was sent.

4. (5 pts) Prove that if $P(X > Y) = 1$ then $E(X) > E(Y)$. Provide a counterexample to show that the converse is not true.
5. Suppose that X and Y are jointly distributed random variables with joint pdf

$$f_{XY}(x, y) = \begin{cases} 2; & 0 < x < y < 1 \\ 0; & \text{elsewhere} \end{cases}$$

- (a) (5 pts) Find the marginal pdf of X .
 - (b) (5 pts) Find the conditional distribution of $Y|X = x$.
 - (c) (5pts) Find the conditional mean and variance of $Y|X = x$.
 - (d) (5 pts) Find $P(Y < 0.5)$.
 - (e) (5 pts) Find $P(Y < 0.5|X = 0.25)$.
 - (f) (10 pts) Let $U = X + Y$. Find the distribution of U . Verify that f_U is a pdf.
6. Let X and Y be independent $N(0, 1)$ random variables.
- (a) (5 pts) What is the distribution of $U = (X - Y)^2/2$? (STOP: Before you dive into a complicated bivariate transformation stop and think about the problem. It is almost literally a one-liner if you appeal to known results.)
 - (b) (3 pts) What is the distribution of $V = (X + Y)^2/2$?