M.S. Numerical Exam 2003

(Department of Mathematical Sciences, M.S.U.)

Instructions: Attempt all questions. Show all work.

1. (a) Let $f(x) = (x+2)(x+1)^2 x(x-1)^3(x-2)$. To which root of f does the Bisection method converge if the initial interval is given by $[-3, -\frac{1}{2}]$?

   (b) Let $f(x)$ denote an arbitrary continuous function with a root $r \in (a, b)$. The Bisection method generates a series of nested intervals $[a, b] = [a_0, b_0], [a_1, b_1], [a_2, b_2], \ldots$ with respective midpoints r_n for $n = 1, 2, 3 \ldots$. Classify the following statements as **Always True** or **Not Always True**. In the case of **Not Always True**, sketch a graph indicating a scenario where the statement is not true.

   - $a_n \leq r \leq r_n$
   - $|r - r_n| \leq |r - r_{n-1}|$
   - $|r - a_n| \leq 2^{-n}(b_0 - a_0)$

2. Consider the tridiagonal structure of the following $n \times n$ linear system.

$$
\begin{array}{rcl}
d_1x_1 + c_1x_2 & = & b_1 \\
a_1x_1 + d_2x_2 + c_2x_3 & = & b_2 \\
a_2x_2 + d_3x_3 + c_3x_4 & = & b_3 \\
\ddots \qquad \vdots & = & \vdots \\
a_{n-2}x_{n-2} + d_{n-1}x_{n-1} + c_{n-1}x_n & = & b_{n-1} \\
a_{n-1}x_{n-1} + d_nx_n & = & b_n
\end{array}
$$

   (a) Give an algorithm for the iterative solution of this system using the Jacobi method. Your algorithm should account for the sparseness of the system matrix.

   (b) Give conditions under which the Jacobi iteration is guaranteed to converge.

   (c) Give the number of multiplications/divisions required for one iteration of the Jacobi method. Also give the number of additions/subtractions.

3. Compute the **local discretization error** for the multistep method given below.

$$y_{k+1} = 4y_k - 3y_{k-1} - 2hf(t_{k-1}, y_{k-1})$$

   Is the method stable? Is the method convergent?

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4. Recall that Simpson's Rule is given by the expression

$$\int_a^b f(x)dx \approx \frac{h}{3}[f(a) + 4f(\frac{a+b}{2}) + f(b)],$$

and the truncation error is given by

$$E(f,h) = \frac{h^5}{90} f^{(4)}(\xi),$$

where $\xi \in (a,b)$ and $h = \frac{b-a}{2}$.

(a) Consider numerical approximation of the integral $\int_0^2 2^x dx$ using Simpson's Rule. Give a geometric interpretation of the quadrature rule.

(b) Approximate the integral using Simpson's Rule.

(c) Give a bound for the amount of truncation error that one can expect from the quadrature rule.