

# Numerical Analysis Master's Exam

January 2006

Be sure to justify your answers for full credit.

1. Compute the LU factorization, without partial pivoting, of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 4 & -1 & 5 \\ -2 & 1 & 1 \end{bmatrix}.$$

Then use this factorization to solve the system  $A\mathbf{x} = \mathbf{b}$  with  $\mathbf{b} = (2, 2, -4)$ .

2. Derive the least squares equations for fitting a line  $y = ax + b$  to data  $(x_i, y_i)$ ,  $i = 1, 2, \dots, N$ .
3. Determine the asymptotic error rate, i.e., the form of the error as  $h \rightarrow 0$ , of the derivative approximation

$$f'(x) \sim \frac{f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)}{12h}$$

4. Suppose the 2-step method

$$y_{n+1} = y_n + \frac{h}{3}f(t_n, y_n) + \frac{2h}{3}f(t_{n+1}, y_{n+1})$$

is applied to the ODE initial value problem

$$\begin{aligned} \frac{dy}{dt} &= -2y, \quad t > 0 \\ y(0) &= y_0 \end{aligned}$$

with fixed step size  $h$ , so  $t_n = nh$  for  $n = 0, 1, \dots$

- (a) For any fixed  $h > 0$ , what is  $\lim_{n \rightarrow \infty} |y_n - y(t_n)|$ ?
- (b) Now fix  $\bar{t} > 0$  and vary both  $h$  and  $\bar{n}$  so that  $\bar{t} = \bar{n}h = t_{\bar{n}}$ . Define the pointwise error  $\bar{e}(h) = y_{\bar{n}} - y(\bar{t})$ . Does  $\lim_{h \rightarrow 0} \bar{e}(h) = 0$ , and if so, what is the asymptotic rate at which the pointwise error goes to zero?

(c) Do you expect the error  $\bar{e}(h)$  to converge to zero *uniformly* for all  $\bar{t} > 0$ ? Explain why or why not.

5. Suppose you are given a new initial value problem (IVP) method, method X, to approximate the solution  $y(1)$  of an IVP  $\dot{y} = f(t, y)$ ,  $y(0) = y_0$ ,  $t \in [0, 1]$ , using various values of (uniform) time step  $h$ . You wish to estimate the order  $\alpha$  of the asymptotic error in  $h$ .

- Describe how you would estimate  $\alpha$  if you know the exact value  $y(1)$ .
- Describe how you would estimate  $\alpha$  if you **don't** know the exact value  $y(1)$ .