2010 Master's Exam on Numerical ODEs

1. Consider the ODE single step method

$$y_n = y_{n-1} + h\, f(t_{n-1}, y_{n-1}) + \frac{h^2}{2} f\left(t_{n-1} + h, y_{n-1} + hf(t_{n-1}, y_{n-1})\right).$$

   a. Using the 2-variable Taylor expansion

$$f(t+h, y+\Delta y) = f(t,y) + \frac{\partial f}{\partial t}(t,y)h + \frac{\partial f}{\partial y}(t,y)\,\Delta y + C_1 h^2 + C_2(\Delta y)^2 + C_3\, h\, \Delta y,$$

   show the local truncation error for this method is $\mathcal{O}(h)$.

   b. Define what it means for a method to be zero-stable and show that this method is zero-stable.

   c. Define what it means for a method to converge. Does this method converge, and if so, what is the order of convergence?

   d. From the standpoint of efficiency, explain why this is a dumb method.

2. Consider the linear multi-step method

$$y_n = y_{n-1} + h\left(\frac{2}{3} f(t_n, y_n) + \frac{1}{3} f(t_{n-1}, y_{n-1})\right). \qquad (1)$$

   a. Find the characteristic polynomial for this method.

   b. Determine whether or not this method is (i) weakly stable, (ii) strongly stable, and/or (iii) zero-stable.

   c. Show that this method is consistent, and determine its order of consistency.

   d. Is this method convergent? If so, what is its order of convergence? Justify your answer.

   d. Find the region of absolute stability for this method.

   e. Is this method stiffly stable? Is it effective for stiff problems? Explain your answer.

   f. This method is *implicit*, since the unknown y_n appears on both sides of the equality in equation (1). Explain how to apply this method to linear, nonhomogeneous systems of the form

$$\mathbf{y}' = A\mathbf{y} + \mathbf{b},$$

   where A is an $n \times n$ matrix and $\mathbf{b}$ is an $n \times 1$ vector with constant coefficients. Supply pseudo-code.

3. Consider the matrix

$$A = \begin{bmatrix} -8 & 3 \\ 6 & 4 \end{bmatrix}$$

   a. Find the 1−norm and ∞−norm of A.

   b. Find A^{-1}, and then find its 1−norm and ∞−norm.

   c. Find 1-norm and ∞-norm bounds on the error in the solution of $A\mathbf{x} = \mathbf{b}$, where the perturbation in $\mathbf{b}$ is $\delta\mathbf{b} = (\epsilon_1, \epsilon_2)^T$.

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

   a. Find the orthogonal projector P (as a 3×3 matrix) onto range(A), and calculate the image under P of the vector $(1, 2, 3)^T$.

   b. Calculate a reduced QR factorization $A = \hat{Q}\hat{R}$ by Gram-Schmidt orthogonalization.

   c. Explain what a least squares solution is. Then use the reduced QR factorization obtained in (b) to find the least squares solution to $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = (1, 2, 3)^T$.

5. Let $v \in \mathbb{R}^m$ be a unit vector.

   a. Give Q, the Householder reflection which reflects across the orthogonal complement of v.

   b. Prove that Q is orthogonal.