

2010 Master's Exam on Numerical ODEs

1. Consider the ODE single step method

$$y_n = y_{n-1} + h f(t_{n-1}, y_{n-1}) + \frac{h^2}{2} f(t_{n-1} + h, y_{n-1} + h f(t_{n-1}, y_{n-1})).$$

a. Using the 2-variable Taylor expansion

$$f(t + h, y + \Delta y) = f(t, y) + \frac{\partial f}{\partial t}(t, y)h + \frac{\partial f}{\partial y}(t, y) \Delta y + C_1 h^2 + C_2 (\Delta y)^2 + C_3 h \Delta y,$$

show the local truncation error for this method is $\mathcal{O}(h)$.

- b. Define what it means for a method to be zero-stable and show that this method is zero-stable.
- c. Define what it means for a method to converge. Does this method converge, and if so, what is the order of convergence?
- d. From the standpoint of efficiency, explain why this is a dumb method.

2. Consider the linear multi-step method

$$y_n = y_{n-1} + h \left(\frac{2}{3} f(t_n, y_n) + \frac{1}{3} f(t_{n-1}, y_{n-1}) \right). \quad (1)$$

- a. Find the characteristic polynomial for this method.
- b. Determine whether or not this method is (i) weakly stable, (ii) strongly stable, and/or (iii) zero-stable.
- c. Show that this method is consistent, and determine its order of consistency.
- d. Is this method convergent? If so, what is its order of convergence? Justify your answer.
- d. Find the region of absolute stability for this method.
- e. Is this method stiffly stable? Is it effective for stiff problems? Explain your answer.
- f. This method is *implicit*, since the unknown y_n appears on both sides of the equality in equation (1). Explain how to apply this method to linear, nonhomogeneous systems of the form

$$\mathbf{y}' = A\mathbf{y} + \mathbf{b},$$

where A is an $n \times n$ matrix and \mathbf{b} is an $n \times 1$ vector with constant coefficients. Supply pseudo-code.

3. Consider the matrix

$$A = \begin{bmatrix} -8 & 3 \\ 6 & 4 \end{bmatrix}$$

- a. Find the 1-norm and ∞ -norm of A .
- b. Find A^{-1} , and then find its 1-norm and ∞ -norm.
- c. Find 1-norm and ∞ -norm bounds on the error in the solution of $A\mathbf{x} = \mathbf{b}$, where the perturbation in \mathbf{b} is $\delta\mathbf{b} = (\epsilon_1, \epsilon_2)^T$.

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- a. Find the orthogonal projector P (as a 3×3 matrix) onto $\text{range}(A)$, and calculate the image under P of the vector $(1, 2, 3)^T$.
- b. Calculate a reduced QR factorization $A = \hat{Q}\hat{R}$ by Gram-Schmidt orthogonalization.
- c. Explain what a least squares solution is. Then use the reduced QR factorization obtained in (b) to find the least squares solution to $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = (1, 2, 3)^T$.

5. Let $v \in \mathbb{R}^m$ be a unit vector.

- a. Give Q , the Householder reflection which reflects across the orthogonal complement of v .
- b. Prove that Q is orthogonal.