Numerical Analysis Master's Exam

January 2011

Instructions: Attempt all questions. Show all work. Carefully read and follow the directions. Clearly label your work and attach it to this sheet.

1. Find constants a, b, c so that

$$F(h) = af(x) + bf(x-h) + cf(x-2h) = f'(x)h + O(h^3)$$

and then use this expression to derive a 2^{nd} order approximation of $f'(x)$.

2. Given the nonlinear function

$$f(x) = \frac{x^2 - 1}{x},$$

use Newton's method to find x_2 (an approximation of the root of $f(x)$ after two iterations) with initial guess $x_0 = 2$. Write your answer in fractions.

3. For the symmetric positive definite matrix

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 6 \end{bmatrix}$$

find the following factorizations of A:

(a) $A = LU$

(b) $A = L^T L$ (Cholesky)

where L and U are lower and upper upper triangular matrices, respectively.

4. An iterative scheme $Qx_{n+1} = (Q - A)x_n + b$ for solving $Ax = b$ has a splitting matrix which uses the backward diagonal elements as in

$$A = \begin{bmatrix} 1/2 & 5 \\ 5 & 1/2 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & 5 \\ 5 & 0 \end{bmatrix}$$

(a) Use the iterative scheme to find a second iterate approximation x_2 of the solution of $Ax = b$ using the initial guess $x_0 = (0,0)^T$ for $b = (50, 50)^T$ and A above.

(b) Find the minimum n value that assures the following relative error tolerance:

$$\frac{||x_{n+1} - x||_1}{||x||_1} < 10^{-6}$$

where $||\cdot||_1$ is the vector 1-norm and x is the exact solution of $Ax = b$.

5. For the linear advection equation with constant velocity $a > 0$ given by

$$u_t + au_x = 0, \quad 0 < x < 1, \quad t > 0,$$

consider a finite difference approximation of the PDE with uniform mesh (space and time step sizes Δx and Δt respectively). Let $U_j^n \approx u(x_j, t_n)$ be the numerical solution at grid point (x_j, t_n).

(a) Give the formula of the upwind scheme. Note here $a > 0$.

(b) What is the CFL condition? Let $\nu = a\Delta t/\Delta x$, for what value of ν does the upwind scheme satisfy the CFL condition?

(c) Find the leading terms of the truncation error of the upwind scheme.

(d) Find the amplification factor $\lambda(k)$ in the Fourier analysis for stability of the upwind scheme.